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Strong-Stability-Preserving, K-Step, 6- to 10-Stage, Hermite–Birkhoff Time-Discretizations of Order 7

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Abstract **– Optimal k-step, 6- to 10-stage, explicit, strong-stability-preserving Hermite–Birkhoff (SSP HB) methods of order 7 with nonnegative coefficients are constructed by combining linear k-step methods of order 4 with 6- to 10-stage Runge–Kutta (RK) methods of order 4. It is seen that the 6-step 6-stage HB method has the largest effective SSP coefficient among the 7th-order HB methods on hand. All new HB methods have larger effective SSP coefficients and larger maximum effective CFL numbers than Huang's hybrid method of order 7, on Burgers' equations, independently of the number k of steps.**

Keywords **– Strong-stability-preserving method; Hermite–Birkhoff method; SSP coefficient; time discretization; method of lines; comparison with other SSP methods.**

I. Introduction

The method of lines is often used to solve time-dependent conservation laws,

$$y_t + g(y)_x = 0, \quad y(x,0) = y_0(x), \tag{1}$$

where the spatial derivative $g(y)_x$ is approximated by a conservative finite difference or finite element at x_j, $j = 1, 2, \ldots, N$, (see, for example, [6], [11], [13], [1]). This spatial semi-discretization produces a system of N ordinary differential equations with initial conditions of the form

$$\frac{dy}{dt} = f(t, y(t)), \qquad y(t_0) = y_0. \tag{2}$$

We say that a time discretization method applied to (2) is strong stability preserving (SSP) in a given norm or seminorm $\|\cdot\|$ if the numerical solution $y_{n,j} \approx y(x_j, t_n)$ satisfies the inequality

$$\|y_{n+1}\| \le \|y_n\|. \tag{3}$$

Natural choices are the total variation semi-norm and the maximum norm.

It is assumed that the first-order forward Euler time discretization, FE,

$$y_{n+1} = y_n + \Delta t f(t_n, y_n), \tag{4}$$

when applied to (2), is SSP for a sufficiently small time step,

$$\Delta t \le \Delta t_{\mathrm{FE}}, \tag{5}$$

dictated by the Courant–Friedrichs–Levy (CFL) condition [2], [9]. This condition restricts the step size ratio $\Delta t/\Delta x$ of a numerical method applied to a hyperbolic partial differential equation such that the domain of dependence of the numerical solution at the point $P = (x_j, t_n)$ contains the domain of dependence of the exact solution at *P*.

In this paper, to solve system (2), we construct new explicit, multistep, multistage, SSP general linear time-discretization methods of order 7 with nonnegative coefficients. These methods, which we call SSP Hermite–Birkhoff (SSP HB) because their construction involves HB interpolation polynomials (see Section II), are combinations of linear k-step methods of order 4 and s-stage RK methods of order 4. The objective of high-order SSP HB time discretizations is to maintain the strong stability property (3) while achieving higher-order accuracy in time, perhaps with a modified CFL restriction, measured here with the SSP coefficient, c(HBks):

$$\Delta t \le c(\mathrm{HB}ks)\Delta t_{\mathrm{FE}}, \tag{6}$$

The SSP coefficient, historically called CFL coefficient, describes the ratio of the strong stability preserving time step to the strongly stable forward Euler time step (see [4]). Since our arguments are based on convex decompositions of high-order methods in terms of the SSP FE method, such high-order methods preserve SSP in any norm once FE is shown to be strongly stable.

Several new explicit 6- to 10-stage SSP HB methods with nonnegative coefficients presented here have been found by computer search.

All new HB methods have larger effective SSP coefficients and larger maximum effective CFL numbers on Burgers' equations, independently of the number of steps, compared to the 7-step Huang hybrid method of order 7 found in [7]. In particular, the 6-step 6-stage HB method has the largest effective SSP coefficient among the 7th-order HB methods on hand.

Section II introduces 6- to 10-stage SSP HB methods. Order conditions are listed in Section III. Section IV derives the Shu–Osher representation of k-step 6- to 10-stage HB methods of order 7. Several new SSP HB methods are constructed in Section V. In Section VI, effective SSP coefficients are compared for several methods. Section VII presents numerical results for several methods applied to Burgers' equations. The new methods are listed in the Appendix in their Shu–Osher form.

II. SSP HB Methods of Order 7 with *k* Steps and *s* Stages

Notation 1: The following notation will be used:

- k denotes the number of steps of a given method,
- s denotes the number of stages of a given method,
- HB*ks* denotes k-step, s-stage SSP Hermite–Birkhoff methods of order 7,
- HM*k* denotes k-step SSP hybrid methods of order 7.

All methods considered in this paper are SSP and of order 7 unless specified otherwise. Therefore the denominations "SSP" and "order 7" will often be omitted in what follows.

Notation 2: The abscissa vector $\sigma = [c_1, c_2, c_3, \ldots, c_s]^T$, $0 \le c_j \le 1$, defines the off-step points $t_n + c_j \Delta t$.

An HB*ks* method requires the following s formulae to perform integration from t_n to t_{n+1}, where, for simplicity, $c_1 = 0$ is used in the summations. By convention, $c_1^0 = 1$.

Let $F_j := f(t_n + c_j\Delta t, Y_j)$ be the jth stage derivative and set $Y_1 = y_n$.

An HB polynomial of degree $2k+i-3$ is used as predictor P_i to obtain the ith stage value Y_i to order 4,

$$Y_i = \sum_{j=0}^{k-1} \alpha_{ij} y_{n-j} + \Delta t \left[\sum_{j=1}^{i-1} a_{ij} F_j + \sum_{j=1}^{k-1} \beta_{ij} f_{n-j} \right], \quad i = 2, 3, \ldots, s. \tag{7}$$

An HB polynomial of degree $2k+s-2$ is used as the integration formula to obtain y_{n+1} to order 7:

$$y_{n+1} = \sum_{j=0}^{k-1} \alpha_j y_{n-j} + \Delta t \left[\sum_{j=1}^{s} b_j F_j + \sum_{j=1}^{k-1} \beta_j f_{n-j} \right]. \tag{8}$$

III. Order Conditions of HB*ks*

To derive the order conditions for HB*ks*, we shall use the following expressions coming from the backsteps of the methods:

$$B_i(j) = \sum_{\ell=1}^{k-1} \alpha_{i\ell} \frac{(-\ell)^j}{j!} + \sum_{\ell=1}^{k-1} \beta_{i\ell} \frac{(-\ell)^{j-1}}{(j-1)!}, \quad j = 1, 2, \ldots, 6, \quad i = 2, 3, \ldots, s. \tag{9}$$

As in the construction of RK methods, we impose the following simplifying conditions on the abscissa vector $\sigma = [c_1, c_2, c_3, \ldots, c_s]^T$:

$$c_i = \sum_{j=1}^{i-1} a_{ij} + B_i(1), \qquad i = 2, 3, \ldots, s. \tag{10}$$

Forcing an expansion of the numerical solution produced by formulae (7)–(8) to agree with a Taylor expansion of the true solution, we obtain multistep- and RK-type order conditions that must be satisfied by HB*ks*. To reduce the large number of RK-type order conditions, we impose the following simplifying assumptions, as in similar searches for ODE solvers [10]:

$$\sum_{j=1}^{i-1} a_{ij} c_j^k + k! B_i(k+1) = \frac{1}{k+1} c_i^{k+1}, \quad \begin{cases} i = 2, 3, \ldots, s, \\ k = 1, 2, 3. \end{cases} \tag{11}$$

Note that (11) with $k = 0$ is (10).

Seven sets of equations remain to be solved:

$$\sum_{j=0}^{k-1} \alpha_{ij} = 1, \quad i = 2, 3, \ldots, s, \tag{12}$$

$$\sum_{i=0}^{k-1} \alpha_i = 1, \tag{13}$$

$$\sum_{i=1}^{s} b_i c_i^k + k! B(k+1) = \frac{1}{k+1}, \quad k = 0, 1, \ldots, 6, \tag{14}$$

$$\sum_{i=2}^{s} b_i \left[\sum_{j=1}^{i-1} a_{ij} \frac{c_j^4}{4!} + B_i(5) \right] + B(6) = \frac{1}{6!}, \tag{15}$$

$$\sum_{i=2}^{s} b_i \frac{c_i}{6} \left[\sum_{j=1}^{i-1} a_{ij} \frac{c_j^4}{4!} + B_i(5) \right] + B(7) = \frac{1}{7!}, \tag{16}$$

$$\sum_{i=2}^{s} b_i \left[\sum_{j=1}^{i-1} a_{ij} \frac{c_j^5}{5!} + B_i(6) \right] + B(7) = \frac{1}{7!}, \tag{17}$$

$$\sum_{i=2}^{s} b_i \left[\sum_{j=1}^{i-1} a_{ij} \left[\sum_{k=1}^{j-1} a_{jk} \frac{c_k^4}{4!} + B_j(5) \right] + B_i(6) \right] + B(7) = \frac{1}{7!}, \tag{18}$$

where the backstep parts, $B(j)$, are defined by

$$B(j) = \sum_{i=1}^{k-1} \alpha_i \frac{(-i)^j}{j!} + \sum_{i=1}^{k-1} \beta_i \frac{(-i)^{j-1}}{(j-1)!}, \quad j = 1, 2, \ldots, 7. \tag{19}$$

IV. Shu–Osher Representation of HB*ks*

We rewrite our s-stage HB*ks* methods in the Shu–Osher representation as convex combinations of FE to show that they satisfy SSP conditions.

Firstly, if we let

$$\lambda_{i\ell} \ge 0, \qquad \sum_{\ell=1}^{i-1} \lambda_{i\ell} = 1, \quad i = 3, 4, \ldots, s, \qquad \text{and} \quad \mu_{s\ell} \ge 0, \quad \sum_{\ell=1}^{s} \mu_{s\ell} = 1, \tag{20}$$

then formulae (7) and (8) become

$$Y_i = \left[\sum_{\ell=1}^{i-1} \lambda_{i\ell}\right] \alpha_{i0} y_n + \sum_{j=1}^{k-1} \alpha_{ij} y_{n-j} + \Delta t \left[\sum_{j=1}^{i-1} a_{ij} F_j + \sum_{j=1}^{k-1} \beta_{ij} f_{n-j}\right], \quad i = 3, 4, \ldots, s, \tag{21}$$

$$y_{n+1} = \left[\sum_{\ell=1}^{s} \mu_{s\ell}\right] \alpha_0 y_n + \sum_{j=1}^{k-1} \alpha_j y_{n-j} + \Delta t \left[\sum_{j=1}^{s} b_j F_j + \sum_{j=1}^{k-1} \beta_j f_{n-j}\right]. \tag{22}$$

Replacing the index *i* by *m* in formula (7), we express y_n as a function of Y_m,

$$y_n = \frac{1}{\alpha_{m0}} \left\{ Y_m - \sum_{j=1}^{k-1} \alpha_{mj} y_{n-j} - \Delta t \left[\sum_{j=1}^{m-1} a_{mj} F_j + \sum_{j=1}^{k-1} \beta_{mj} f_{n-j}\right]\right\}, \quad m = 2, 3, \ldots, s. \tag{23}$$

For $i = 3, 4, \ldots, s$ and $\ell = 1, 2, \ldots, i-1,$, we replace the variable y_n in the terms $\lambda_{i\ell}\alpha_{i0}y_n$ in (21) by the right-hand side of (23) with $m = \ell$. Similarly, y_n in the terms $\mu_{s\ell}\alpha_0 y_n$ in (22) is replaced by the right-hand sides of (23) with $m = \ell$.

Secondly, we rewrite (7) with *i* = 2, and (21) with $i = 3, 4, \ldots, s$ as (24), and (22) as (25) in the Shu–Osher equivalent form:

$$Y_i = \left[\sum_{j=0}^{k-1} A_{ij} y_{n-j} + \Delta t \sum_{j=0}^{k-1} B_{ij} f_{n-j}\right] \left[\sum_{j=2}^{i-1} e_{ij} Y_j + \Delta t \sum_{j=2}^{i-1} g_{ij} F_j\right], \quad i = 2, 3, \ldots, s, \tag{24}$$

$$y_{n+1} = \left[\sum_{j=0}^{k-1} A_j y_{n-j} + \Delta t \sum_{j=0}^{k-1} B_j f_{n-j}\right] + \left[\sum_{j=2}^{s} e_j Y_j + \Delta t \sum_{j=2}^{s} g_j F_j\right], \tag{25}$$

where the coefficients are

$$A_{ij} = \alpha_{i,j} - \sum_{\ell=2}^{i-1} e_{i\ell}\alpha_{\ell j}, \quad j = 0, 1, \ldots, k-1, \; i = 2, 3, \ldots, s,$$

$$A_j = \alpha_j - \sum_{\ell=2}^{s} e_\ell \alpha_{\ell j}, \quad j = 0, 1, \ldots, k-1,$$

$$B_{ij} = \beta_{i,j} - \sum_{\ell=2}^{i-1} e_{i\ell}\beta_{\ell j}, \quad j = 0, 1, \ldots, k-1, \; i = 2, 3, \ldots, s,$$

$$B_j = \beta_j - \sum_{\ell=2}^{s} e_\ell \beta_{\ell j}, \quad j = 0, 1, \ldots, k-1,$$

$$g_{ij} = a_{ij} - \sum_{\ell=j+1}^{i-1} e_{i\ell} a_{\ell j}, \; i = 3, 4, \ldots, s, \; j = 2, 3, \ldots, i-1,$$

$$g_j = b_j - \sum_{\ell=j+1}^{s} e_\ell a_{\ell j}, \quad j = 2, 3, \ldots, s,$$

which follow from setting

$$\begin{aligned} e_{ij} &= \lambda_{ij}\alpha_{i0}/\alpha_{j0}, \quad && j = 2, 3, \ldots, i-1, \quad i = 3, 4, \ldots, s, \\ a_{i1} &= \beta_{i0}, && i = 2, 3, \ldots, s, \\ e_j &= \mu_{sj}\alpha_0/\alpha_{j0}, && j = 2, 3, \ldots, s, \\ b_1 &= \beta_0. \end{aligned}$$

Thirdly, the representation (24)–(25), under the assumptions that all coefficients are nonnegative, implies that the HB*kp* are SSP. In fact, one finds that the following functions are convex combinations of forward Euler steps:

- In (24) for $i = 2, 3, \ldots, s,$ the first and second bracketed terms are sums of FE steps with step sizes $\frac{B_{ij}}{A_{ij}}\Delta t,$ $j = 0, \ldots, k-1,$ and $\frac{g_{ij}}{e_{ij}}\Delta t,$ $j = 2, 3, \ldots, i-1,$ respectively.
- In (25), the first and second bracketed terms are sums of FE steps with step sizes $\frac{B_j}{A_j}\Delta t,$ $j = 0, \ldots, k-1,$ and $\frac{g_j}{e_j}\Delta t,$ $j = 2, 3, \ldots, s,$ respectively.

One can easily verify that

$$\sum_{j=0}^{k-1} A_{ij} + \sum_{j=2}^{i-1} e_{ij} = 1, \; i = 2, 3, \ldots, s, \quad \sum_{j=0}^{k-1} A_j + \sum_{j=2}^{s} e_j = 1.$$

Provided all the coefficients A_{ij}, e_{ij}, A_j, e_j are nonnegative, the following straightforward extension of a result presented in [5], [7] holds.

Theorem 1: If the forward Euler method FE is SSP under the CFL condition $\Delta t \le \Delta t_{\mathrm{FE}},$ then the *k*-step, *s*-stage HB*ks* methods (24)–(25) are SSP provided

$$\Delta t \le c(\mathrm{HB}ks)\Delta t_{\mathrm{FE}},$$

where the SSP coefficient *c*(HB*ks*) is the minimum of the four numbers:

$$\min_{j=0,1,\ldots,k-1}\left\{\frac{A_j}{B_j}\right\},\quad \min_{j=0,1,\ldots,k-1}\left\{\frac{A_{ij}}{B_{ij}}\right\},\ i=2,3,\ldots,s,$$
$$\min_{j=2,3,\ldots,s}\left\{\frac{e_j}{g_j}\right\},\quad \min_{j=2,3,\ldots,i-1}\left\{\frac{e_{ij}}{g_{ij}}\right\},\ i=3,4,\ldots,s, \tag{26}$$

with the convention that $a/0 = +\infty$, under the assumption that all coefficients of (24)–(25) are nonnegative.

V. Construction of Optimal HB*ks*

Since HB*ks* contain many free parameters when k is sufficiently large, we use the Matlab Optimization Toolbox to search for the methods with the largest c(HB*ks*) for different k and s. To optimize HB*ks*, we maximize c(HB*ks*) of Theorem 1 by solving the nonlinear programming problem

$$\max_{A_{ij},B_{ij},e_{ij},g_{ij},A_j,B_j,e_j,g_j} c(\mathrm{HB}ks), \tag{27}$$

where all the numbers in all pairs

$$\begin{aligned}
&\{A_j, B_j\}, \quad j = 0,1,\ldots,k-1,\\
&\{A_{ij}, B_{ij}\}, \quad i = 2,3,\ldots,s, \quad j = 0,1,\ldots,k-1,\\
&\{e_j, g_j\}, \quad j = 2,3,\ldots,s,\\
&\{e_{ij}, g_{ij}\}, \quad i = 3,4,\ldots,s, \quad j = 2,3,\ldots,i-1,
\end{aligned}$$

are nonnegative. Null pairs, {0, 0}, are not included in the minimization process if they occur. Besides the nonnegativity constraints on all variables, the objective function (27) is subject to

- the convex combination constraints (20),
- the simplifying assumptions (10) and (11) for HB*ks*,
- the order conditions for HB*ks* (12) to (18),
- the conditions on the abscissae c_i: $c_1 = 0$, $0 \le c_i \le 1$, $i = 2, 3, \ldots, s$.

VI. Comparing Effective SSP Coefficients

Definition 1: (See [12]) The *effective SSP coefficients* of the SSP method M is denoted by

$$c_{\mathrm{eff}}(M) = \frac{c(M)}{\ell}, \tag{28}$$

where ℓ is the number of function evaluations of method M per time step and $c(M)$ is the SSP coefficient of M.

The SSP coefficients, c(HM), of hybrid methods are defined in [7]. In this paper, $\ell = 4, 5, \ldots, 10$ for HB methods and $\ell = 2$ for hybrid methods. The numbers $c_{\mathrm{eff}}(\mathrm{HB})$ and $c_{\mathrm{eff}}(\mathrm{HM})$ will be used below.

The effective SSP coefficients, c_{eff}, provide a fair comparison between methods of the same order, although, in practice, starting methods and storage issues may also be important. Gottlieb [3] pointed out that one looks for high-order SSP methods M with $c(M)$ as large as possible, taking their computational costs and orders into account.

Huang [7] introduced a 7-step HM7 with c(HM7) = 0.234 and c_{eff}(HM7) = 0.117. We now list our best methods.

HB*k*4. Our best 4-stage SSP HB*k*4 methods have step number k = 4, 5, 6, 7. We remark that, even with a small number of stages, these new methods are competitive with Huang's 7-step HM7. For example, c_{eff}(HB44) = 0.141 > c_{eff}(HM7) = 0.117, where HB44 uses only four steps.

HB*k*5. For 5-stage methods, we found HB*k*5 with k = 3, 4, 5, 6, 7 and c_{eff}(HB*k*5) > c_{eff}(HB*k*4) for k = 4, 5, 6, 7. Our best HB*k*5 is HB75 with c_{eff}(HB75) = 0.296. We remark that increasing k or s in HB*ks* improves c_{eff}(HB*ks*) as when increasing the number of stages in Runge–Kutta methods of lower order.

HB*k*6. For 6-stage methods, we found HB*k*6 with k = 3, 4, 5, 6, 7, one of which requires only k = 3. Moreover, we found HB66 with the largest effective SSP coefficient among the seventh-order HB methods on hand. Here $c(\mathrm{HB66}) = 1.828$, $c_{\mathrm{eff}}(\mathrm{HB66}) = 0.305$ listed in boldface in Table I. According to our search, it seems that c_{eff}(HB66) cannot be improved up to 10 stages and 7 steps.

The formulae of the new HB*ks* are listed in Appendix with their c(HB*ks*), c_{eff}(HB*ks*) and abscissa vector σ.

TABLE I

$c_{\mathrm{EFF}}(\mathrm{HB}ks)$, FOR K = 3, 4, . . . , 7, AS A FUNCTION OF S

	$c_{\mathrm{eff}}(\mathrm{HB}ks)$				
$s\backslash k$	3	4	5	6	7
4		0.141	0.219	0.256	0.287
5	0.173	0.239	0.282	0.293	0.296
6	0.232	0.290	0.301	**0.305**	0.305
7	0.231	0.286	0.292	0.293	0.293
8	0.228	0.280	0.285	0.285	0.285
9	0.209	0.262	0.277	0.277	0.277
10	0.191	0.240	0.255	0.260	0.260

TABLE II

$\mathrm{PEG}(c_{\mathrm{EFF}}(\mathrm{HB}ks), c_{\mathrm{EFF}}(\mathrm{HM7}))$ AS A FUNCTION OF S WHERE c(HM7) = 0.234 AND $c_{\mathrm{EFF}}(\mathrm{HM7}) = 0.117$. COMPARISON IS ROW-WISE

s	HB*ks*	c(HB*ks*)	c_{eff}(HB*ks*)	PEG
4	HB44	0.564	0.141	21 %
	HB54	0.877	0.219	87 %
	HB64	1.023	0.256	119 %
	HB74	1.148	0.287	145 %
5	HB35	0.868	0.173	48 %
	HB45	1.197	0.239	105 %
	HB55	1.410	0.282	141 %
	HB65	1.463	0.293	150 %
	HB75	1.481	0.296	153 %
6	HB36	1.395	0.232	99 %
	HB46	1.739	0.290	148 %
	HB56	1.804	0.301	157 %
	HB66	**1.828**	**0.305**	160 %
7	HB37	1.619	0.231	98 %
	HB47	2.000	0.286	144 %
	HB57	2.046	0.292	150 %

Table I lists $c_{\text{eff}}(\text{HB}ks)$ as a function of s for k = 3, 4, . . . , 7. We note that, for a given k, $c_{\text{eff}}(\text{HB}ks)$ first increases with s and then decreases. On the other hand, for a given s, $c_{\text{eff}}(\text{HB}ks)$ first increases with k and then stabilizes.

Definition 2: The *percentage efficiency gain* (PEG) of the SSP coefficients $c_{\text{eff}}(\text{M2})$ of method 2 over $c_{\text{eff}}(\text{M1})$ of method 1 is

$$\text{PEG}(c_{\text{eff}}(\text{M2}), c_{\text{eff}}(\text{M1})) = \frac{c_{\text{eff}}(\text{M2}) - c_{\text{eff}}(\text{M1})}{c_{\text{eff}}(\text{M1})}. \quad (29)$$

Table II lists $c(\text{HB}ks)$, and compares $c_{\text{eff}}(\text{HB}ks)$ and $c_{\text{eff}}(\text{HM7})$ by the $\text{PEG}(c_{\text{eff}}(\text{HB}ks), c_{\text{eff}}(\text{HM7}))$ where $c(\text{HM7}) = 0.234$ and $c_{\text{eff}}(\text{HM7}) = 0.117$. Comparison is row-wise. It is seen that $c_{\text{eff}}(\text{HB}ks) > c_{\text{eff}}(\text{HM7})$, for all values of k and s on hand.

In Fig. 1, HBk6 and HM7 both of order 4, and HM of order 6 of Huang [7] are compared on the basis of their effective SSP coefficients as a function of the number of steps, k. It is seen that HBk6 have larger effective SSP coefficients than HM for k = 5, 6, 7.

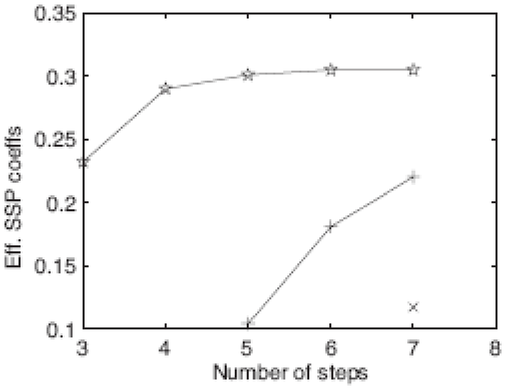

Fig. 1. Effective SSP coefficients versus number of steps k of the following methods: 6-stage HBk6 of order 7, HM7 of order 7 ×, HMk of order 6 +

VII. Numerical Results

From now on, we use the total variation semi-norm,

$$\text{TV}(y_n) = \sum_j |y_{n,j+1} - y_{n,j}|, \quad (30)$$

and say that a method is total variation diminishing (TVD) if

$$\text{TV}(y_{n+1}) \leq \text{TV}(y_n). \quad (31)$$

We compare our new methods numerically with Huang's HM of order 7.

A. Starting Procedure

To maintain the TVD property (31), the necessary starting values for HBkp were obtained by RK54 with small initial step size, $h_1 = 1.0\,\text{e-}04$ (approximatively).

B. Comparing HBks with Other Methods on Burgers' Equation with a Unit Downstep Initial Condition

As a first comparison of HBks with HM7, following Huang [7], we consider Burgers' equation in Problem 1.

Problem 1: Burgers' equation with a unit downstep initial condition:

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left[\frac{1}{2}u(x,t)^2\right] = 0, \qquad u(x,0) = \begin{cases} 1, & -1 \leq x < 0, \\ 0, & 0 < x \leq 1. \end{cases} \quad (32)$$

and boundary condition $u(-1,t) = 1$ for $t \geq 0$.

We discretize the spatial derivative of the flux function $f(u) = u(x,t)^2/2$ by the weighted essentially non-oscillatory finite difference scheme of order 5 (WENO5) of Jiang and Shu [8] with spatial stepsize $\Delta x = 1/150$. This leads to the semi-discrete system

$$\frac{d}{dt}u_j(t) = -\frac{1}{\Delta x}\left[f_{j+(1/2)} - f_{j-(1/2)}\right], \quad (33)$$

where $u_j(t) \approx u(x_j, t)$ with $x_j = j\Delta x$, j = ... ,−2,−1, 0, 1, 2, . . ., and $f_{j+(1/2)}$ is the numerical flux, which typically is a Lipschitz continuous function of several neighboring values $u_j(t)$ (see [8] for details). A time discretization can then be applied to (33).

We consider the total variation norm of the numerical solution at $t_{\text{final}} = 1.8$. For this purpose, we let num_{eff} be the *largest effective CFL number* defined as

$$\text{num}_{\text{eff}} = \max_{\Delta t}\left\{\frac{\Delta t}{\Delta x}\frac{1}{\ell}\right\}, \quad (34)$$

such that the TV error in the numerical solution satisfies the inequality

$$|\,\text{TV}(u(x,t_{\text{final}})) - \text{TV}(u(x,0))| \leq 5.0\,\text{e-}02, \quad (35)$$

and we let $\max \Delta t_{\text{num}} = \ell \Delta x\, \text{num}_{\text{eff}}$ be the maximum numerical step size. Here ℓ is the number of function evaluations per time step. We note that inequality (35) is used because t_{final} is small.

Finally, we let $\max \Delta t_{\text{theor}}$ of HBks for problem 1 be taken as

$$\max \Delta t_{\text{theor}} = c(\text{HB}ks)\Delta t_{\text{FE}}, \quad (36)$$

where the SSP coefficient, $c(\text{HB}ks)$, of each HBks is listed in Appendix.

The numerical results for Problem 1 show that the forward Euler method, FE, has TVD property (31) with error (35) under the time step restriction

$$\Delta t \leq \Delta t_{\text{FE}} = \text{num}_{\text{eff}}(\text{FE})\Delta x = 0.325\Delta x. \quad (37)$$

It was also observed numerically that the TVD property (31) holds with error (35) for the methods listed in Table I with $\Delta t \leq \max \Delta t_{\mathrm{num}}$. This confirms the result of Theorem 1 that HB*ks* are also TVD when combined with the WENO5 space discretization since HB*ks* are convex combinations of FE. The same situation holds for Problem 2 below.

Definition 3: The $\mathrm{num}_{\mathrm{eff}}$ *percentage efficiency gain* of methods M2 over M1 is

$$\mathrm{PEG}(\mathrm{num}_{\mathrm{eff}}) = \frac{\mathrm{num}_{\mathrm{eff}}(\mathrm{M2}) - \mathrm{num}_{\mathrm{eff}}(\mathrm{M1})}{\mathrm{num}_{\mathrm{eff}}(\mathrm{M1})}. \quad (38)$$

We shall use the ratio $R_{\mathrm{n/t}} = \max \Delta t_{\mathrm{num}} / \max \Delta t_{\mathrm{theor}}$ for HB*ks* and HM7.

Table III lists results for Problem 1. The $\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$ is listed in columns 3 as a function of *s*. The $\mathrm{PEG}(\mathrm{num}_{\mathrm{eff}})$ of HB*ks* over HM7 are in column 5. In this case, $\mathrm{num}_{\mathrm{eff}}(\mathrm{HM7}) = 0.127$ and $R_{\mathrm{n/t}}(\mathrm{HM7}) = 3.340$.

It is seen that

(a) $\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks) > \mathrm{num}_{\mathrm{eff}}(\mathrm{HM7})$ for all HB*ks* on hand,

(b) generally, $R_{\mathrm{n/t}}(\mathrm{HB}kp)$ increases as *k* decreases,

(c) $\mathrm{PEG}(\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks), \mathrm{num}_{\mathrm{eff}}(\mathrm{HM7})) > 0$ and increases as $s \leq 6$ increases.

Table IV lists the $\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$, for *k* = 3, 4, . . . , 7, as a function of *s* for Problem 1.

TABLE III

$\mathrm{PEG}=\mathrm{PEG}(\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HB}ks), \mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HM7}))$ WHERE $\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HM7}) = 0.127$ AND $R_{\mathrm{N/T}}(\mathrm{HM7}) = 3.340$ FOR PROBLEM 1

s	HBks	$\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$	$R_{\mathrm{n/t}}(\mathrm{HB}ks)$	PEG
4	HB44	0.199	4.343	57 %
	HB54	0.204	2.863	61 %
	HB64	0.249	2.996	96 %
	HB74	0.234	2.509	84 %
5	HB35	0.274	4.858	116 %
	HB45	0.279	3.586	120 %
	HB55	0.294	3.208	131 %
	HB65	0.304	3.197	139 %
	HB75	0.309	3.210	143 %
6	HB36	0.219	2.898	72 %
	HB46	0.244	2.590	92 %
	HB56	0.294	3.009	131 %
	HB66	0.269	2.717	112 %

TABLE IV

$\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HB}ks)$, FOR *k* = 3, 4, . . . , 7, AS A FUNCTION OF *S* APPLIED TO PROBLEM 1

	$\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$				
$s\backslash k$	3	4	5	6	7
4		0.199	0.204	0.249	0.234
5	0.274	0.279	0.294	0.304	0.309
6	0.219	0.244	0.294	0.269	0.269

C. Comparing HBks and Other Methods on Burgers' Equation with a Square-Wave Initial Condition

As a second comparison, we consider Burgers' equation with a square-wave initial value in Problem 2, which is one of Laney's five test problems [9].

Problem 2: Burgers' equation with a square wave initial condition:

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left[\frac{1}{2}\,u(x,t)^2\right] = 0, \qquad u(x,0) = \begin{cases} 1, & |x| \leq \frac{1}{3}, \\ 0, & \frac{1}{3} < |x| \leq 1. \end{cases} \quad (39)$$

and boundary condition $u(-1,t) = 0$ for $t \geq 0$.
We discretize the spatial derivative of Problem 2 by WENO5 and compute the total variation of the numerical solution as a function of the effective CFL number, $\Delta t/(\ell \Delta x)$, at $t_{\mathrm{final}} = 0.6$. In this case, $\mathrm{num}_{\mathrm{eff}}(\mathrm{FE}) = 0.325$ in the time step restriction (37) is replaced by $\mathrm{num}_{\mathrm{eff}}(\mathrm{FE}) = 0.183$.

The $\mathrm{num}_{\mathrm{eff}}$ of HB*ks* applied to Problem 2 are listed in columns 3 of Table V. Column 5 lists the $\mathrm{PEG}(\mathrm{num}_{\mathrm{eff}})$ of HB methods over HM7.

TABLE V

$\mathrm{PEG}=\mathrm{PEG}(\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HB}ks), \mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HM7}))$ WHERE $\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HM7}) = 0.122$ AND $R_{\mathrm{N/T}}(\mathrm{HM7}) = 5.689$ FOR PROBLEM 2

s	HBks	$\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$	$R_{\mathrm{n/t}}(\mathrm{HB}ks)$	PEG
4	HB44	0.209	8.087	71 %
	HB54	0.209	5.200	71 %
	HB64	0.239	5.098	96 %
	HB74	0.254	4.828	108 %
5	HB35	0.269	8.457	120 %
	HB45	0.269	6.130	120 %
	HB55	0.294	5.688	141 %
	HB65	0.314	5.854	157 %
	HB75	0.284	5.231	133 %
6	HB36	0.214	5.021	75 %
	HB46	0.259	5.425	112 %
	HB56	0.299	5.425	145 %
	HB66	0.269	4.817	120 %

TABLE VI

$\mathrm{NUM}_{\mathrm{EFF}}(\mathrm{HB}ks)$, FOR *K* = 3, 4, . . . , 7, AS A FUNCTION OF *S* APPLIED TO PROBLEM 2

	$\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$				
$s\backslash k$	3	4	5	6	7
4		0.209	0.209	0.239	0.254
5	0.269	0.269	0.294	0.314	0.284
6	0.214	0.259	0.299	0.269	0.269

It is seen that the results for Problem 2 listed in Table V confirm the observations (a)–(c) obtained for Problem 1 as listed in Table III.

We observe that, as with hybrid methods, the ratio $\max \Delta t_{\mathrm{num}} / \max \Delta t_{\mathrm{theor}}$ of HB*ks* for Problems 1 and 2 are greater than 1. The theoretical strong stability bounds of HB*ks* are then verified in the numerical comparison of maximum time steps for Problem 1 and confirmed again with Problem 2.

Table VI lists the $\mathrm{num}_{\mathrm{eff}}(\mathrm{HB}ks)$ for k = 3, 4, . . . , 7 as a function s for Problem 2.

VIII. Conclusion

New optimal explicit k-step 6- to 10-stage s of SSP Hermite Birkhoff methods, HBks, of orders 7 with nonnegative coefficients are constructed by combining linear k-step methods of order 4 with 6- to 10-stage Runge–Kutta methods of order 4. In particular, the 6-step 6-stage HB66 has the largest effective SSP coefficient among the seventh-order HB methods on hand. Compared to Huang's hybrid method HM7 of order 7, all new HBks have larger effective SSP coefficients and larger maximum effective CFL numbers on Burgers' equations, independently of the number of steps.

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Appendix

This appendix lists the Shu–Osher representation of the HBks methods considered in this paper with their $c(\mathrm{HB}ks)$, $c_{\mathrm{eff}}(\mathrm{HBks})$ and abscissa vector $\sigma = [c_1, c_2, \ldots]$.

HB44. Here $c = 0.564$, $c_{\mathrm{eff}} = 0.141$, and $\sigma = [0, 0.47253084193946687, 0.62298379770112466, 0.77343675346274787]^T$.

$$\begin{aligned}
Y_2 &= 1.4181171301608114\,\text{e-01}\,y_{n-3} + 2.6746986944809459\,\text{e-01}\,y_{n-2} \\
&+ 8.8729483281577440\,\text{e-02}\,y_{n-1} + 5.0198893425424684\,\text{e-01}\,y_n \\
&+ 4.7424507571486046\,\text{e-01}\,\Delta t f_{n-2} + 1.5732433938761134\,\text{e-01}\,\Delta t f_{n-1} \\
&+ 8.9006578806300507\,\text{e-01}\,\Delta t f_n,
\end{aligned}$$

$$\begin{aligned}
Y_3 &= 5.6455583476077723\,\text{e-02}\,y_{n-3} + 1.7475976037994695\,\text{e-01}\,y_{n-2} \\
&+ 3.7086510533962463\,\text{e-01}\,y_{n-1}\, 1.8183631650368465\,\text{e-01}\,y_n \\
&+ 4.7513634361493660\,\text{e-02}\,\Delta t f_{n-3} + 6.5757294578528802\,\text{e-01}\,\Delta t f_{n-1} \\
&+ 3.2241006385482263\,\text{e-01}\,\Delta t f_n + 2.1608323430066603\,\text{e-01}\,Y_2 \\
&+ 3.8313253759417537\,\text{e-01}\,\Delta t F_2,
\end{aligned}$$

$$\begin{aligned}
Y_4 &= 2.4221142166745917\,\text{e-02}\,y_{n-3} + 4.8697070969724393\,\text{e-02}\,y_{n-2} \\
&+ 1.9256679584245262\,\text{e-12}\,y_{n-1} + 7.1071176389436097\,\text{e-01}\,y_n \\
&+ 8.6343729694796853\,\text{e-02}\,\Delta t f_{n-2} + 3.4143798899322064\,\text{e-12}\,\Delta t f_{n-1} \\
&+ 3.6397677596282374\,\text{e-01}\,\Delta t f_n + 1.3133260011104431\,\text{e-02}\,Y_2 \\
&+ 2.0323676295613866\,\text{e-01}\,Y_3 + 3.6035473541396956\,\text{e-01}\,\Delta t F_3,
\end{aligned}$$

$$\begin{aligned}
y_{n+1} &= 3.3921902937070918\,\text{e-02}\,y_{n-3} + 2.5714116547015001\,\text{e-02}\,y_{n-2} \\
&+ 3.9501177301951948\,\text{e-02}\,y_{n-1} + 5.3391509806078963\,\text{e-01}\,y_n \\
&+ 1.3755767756843626\,\text{e-02}\,\Delta t f_{n-3} + 4.5593147272786197\,\text{e-02}\,\Delta t f_{n-2} \\
&+ 7.0038688316725384\,\text{e-02}\,\Delta t f_{n-1} + 3.0756492028491300\,\text{e-01}\,\Delta t f_n \\
&+ 8.6160403455941698\,\text{e-02}\,Y_2 + 2.8078730169723076\,\text{e-01}\,Y_4 \\
&+ 4.9785792855077493\,\text{e-01}\,\Delta t F_4.
\end{aligned}$$

HB54. Here $c = 0.877$, $c_{\mathrm{eff}} = 0.219$, and $\sigma = [0, 0.24952030418221310, 0.53315370185373812, 0.81678709952540329]^T$.

$$\begin{aligned}
Y_2 &= 2.3982101438859836\,\text{e-02}\,y_{n-4} + 8.7678139889331416\,\text{e-02}\,y_{n-3} \\
&+ 5.3701393015570177\,\text{e-02}\,y_{n-2} + 3.8841208288909290\,\text{e-01}\,y_{n-1} \\
&+ 4.4622628276714571\,\text{e-01}\,y_n + 9.9887959665517417\,\text{e-02}\,\Delta t f_{n-3} \\
&+ 6.1179703245210756\,\text{e-02}\,\Delta t f_{n-2} + 4.3486379190199109\,\text{e-01}\,\Delta t f_{n-1} \\
&+ 5.0836654371316070\,\text{e-01}\,\Delta t f_n,
\end{aligned}$$

$$\begin{aligned}
Y_3 &= 1.1796058355794571\,\text{e-02}\,y_{n-4} + 8.3827812174208732\,\text{e-02}\,y_{n-3} \\
&+ 4.0125168106181686\,\text{e-01}\,y_{n-1} + 7.7216192941462425\,\text{e-02}\,\Delta t f_{n-3} \\
&+ 4.5712890104892312\,\text{e-01}\,\Delta t f_{n-1} + 5.0312444840817983\,\text{e-01}\,Y_2 \\
&+ 5.7318819346265659\,\text{e-01}\,\Delta t F_2,
\end{aligned}$$

$$\begin{aligned}
Y_4 &= 1.4755308341935032\,\text{e-02}\,y_{n-4} + 6.1776665970132170\,\text{e-02}\,y_{n-3} \\
&+ 3.0637732023705277\,\text{e-01}\,y_{n-1} + 1.5743252991886703\,\text{e-01}\,y_n \\
&+ 7.0379516792679092\,\text{e-02}\,\Delta t f_{n-3} + 3.4904259425320944\,\text{e-01}\,\Delta t f_{n-1} \\
&+ 1.7935615671620367\,\text{e-01}\,\Delta t f_n + 4.5965817553201299\,\text{e-01}\,Y_3 \\
&+ 5.2366892540627263\,\text{e-01}\,\Delta t F_3,
\end{aligned}$$

$$\begin{aligned}
y_{n+1} &= 1.1895330671295582\,\text{e-02}\,y_{n-4} + 7.4076433114412518\,\text{e-02}\,y_{n-2} \\
&+ 1.2066305045361436\,\text{e-02}\,y_{n-1} + 5.4016773987737765\,\text{e-01}\,y_n \\
&+ 6.0405726702878262\,\text{e-03}\,\Delta t f_{n-4} + 8.4392116124240218\,\text{e-02}\,\Delta t f_{n-2} \\
&+ 1.3746625934402956\,\text{e-02}\,\Delta t f_{n-1} + 3.9593568274754259\,\text{e-01}\,\Delta t f_n \\
&+ 3.6179419129155271\,\text{e-01}\,Y_4 + 4.1217666835272870\,\text{e-01}\,\Delta t F_4.
\end{aligned}$$

HB64. Here $c = 1.023$, $c_{\mathrm{eff}} = 0.255$, and $\sigma = [0, 0.29466804302887456, 0.53712151898473259, 0.82017022726301958]^T$.

$$\begin{aligned}
Y_2 &= 5.6855872260172732\,\text{e-03}\,y_{n-5} + 7.9297284979339799\,\text{e-02}\,y_{n-3} \\
&+ 4.3500789258590154\,\text{e-02}\,y_{n-2} + 3.2326556643968146\,\text{e-01}\,y_{n-1} \\
&+ 5.4825077209637141\,\text{e-01}\,y_n + 7.7458278661341431\,\text{e-02}\,\Delta t f_{n-3} \\
&+ 4.2491949847438738\,\text{e-02}\,\Delta t f_{n-2} + 3.1576862099913094\,\text{e-01}\,\Delta t f_{n-1} \\
&+ 5.3553612954593099\,\text{e-01}\,\Delta t f_n,
\end{aligned}$$

$$\begin{aligned}
Y_3 &= 2.0822724469317017\,\text{e-03}\,y_{n-5} + 3.2215004365335170\,\text{e-02}\,y_{n-4} \\
&+ 9.2042336478407524\,\text{e-03}\,y_{n-3} + 4.2684197678291180\,\text{e-01}\,y_{n-1} \\
&+ 3.1467896862504119\,\text{e-02}\,\Delta t f_{n-4} + 8.9907755977306839\,\text{e-03}\,\Delta t f_{n-3} \\
&+ 4.1694296079144122\,\text{e-01}\,\Delta t f_{n-1} + 5.2965651275698045\,\text{e-01}\,Y_2 \\
&+ 5.1737309506389240\,\text{e-01}\,\Delta t F_2,
\end{aligned}$$

$$
\begin{aligned}
Y_4 &= 3.3380614950874847\,\text{e-03} y_{n-5} + 5.0440369871285376\,\text{e-02} y_{n-3} \\
&+ 2.5660915848408855\,\text{e-01} y_{n-1} + 1.8461916259929675\,\text{e-01} y_n \\
&+ 4.9270592634906657\,\text{e-02} \Delta t f_{n-3} + 2.5065806111888317\,\text{e-01} \Delta t f_{n-1} \\
&+ 1.8033760609289001\,\text{e-01} \Delta t f_n + 5.0499324755024178\,\text{e-01} Y_3 \\
&+ 4.9328180278850281\,\text{e-01} \Delta t F_3, \\
y_{n+1} &= 2.9026243385924483\,\text{e-03} y_{n-5} + 4.0701406471331039\,\text{e-03} y_{n-3} \\
&+ 4.9510098281349141\,\text{e-02} y_{n-2} + 5.2856101283494494\,\text{e-01} y_n \\
&+ 2.0405771171904080\,\text{e-03} \Delta t f_{n-5} + 3.9757488357719736\,\text{e-03} \Delta t f_{n-3} \\
&+ 4.8361895243025151\,\text{e-02} \Delta t f_{n-2} + 3.3011253799304868\,\text{e-01} \Delta t f_n \\
&+ 8.4004096111150329\,\text{e-03} Y_2 + 4.0655571428686538\,\text{e-01} Y_4 \\
&+ 8.2055932772060330\,\text{e-03} \Delta t F_2 + 3.9712716288833166\,\text{e-01} \Delta t F_4.
\end{aligned}
$$

HB74. Here $c = 1.148$, $c_{\text{eff}} = 0.287$, and
$\sigma = [0, 0.25551320068703615, 0.49257818032067280, 0.84767837996907580]^T$.

$$
\begin{aligned}
Y_2 &= 1.4810747800482122\,\text{e-03} y_{n-6} + 6.2117213460275887\,\text{e-04} y_{n-4} \\
&+ 7.4054828988991600\,\text{e-02} y_{n-3} + 4.1606073711662151\,\text{e-03} y_{n-2} \\
&+ 3.7182580251349062\,\text{e-01} y_{n-1} + 5.4785651421170045\,\text{e-01} y_n \\
&+ 5.4072108500486532\,\text{e-04} \Delta t f_{n-4} + 6.4463624895834862\,\text{e-02} \Delta t f_{n-3} \\
&+ 3.6217467054521736\,\text{e-03} \Delta t f_{n-2} + 3.2366881926613467\,\text{e-01} \Delta t f_{n-1} \\
&+ 4.7690093017610774\,\text{e-01} \Delta t f_n, \\
Y_3 &= 7.8285481545311750\,\text{e-05} y_{n-6} + 2.8227209326948936\,\text{e-02} y_{n-4} \\
&+ 4.0839155782542635\,\text{e-01} y_{n-1} + 2.4571365010904813\,\text{e-02} \Delta t f_{n-4} \\
&+ 3.5549876427636362\,\text{e-01} \Delta t f_{n-1} + 5.6330294736607944\,\text{e-01} Y_2 \\
&+ 4.9034682001795038\,\text{e-01} \Delta t F_2, \\
Y_4 &= 7.1704107456275394\,\text{e-04} y_{n-6} + 6.1235432988951365\,\text{e-03} y_{n-4} \\
&+ 2.0048927610024553\,\text{e-02} y_{n-3} + 2.7641589466966882\,\text{e-01} y_{n-1} \\
&+ 5.3304531742563235\,\text{e-03} \Delta t f_{n-4} + 1.7452292668294348\,\text{e-02} \Delta t f_{n-3} \\
&+ 2.4061591651073727\,\text{e-01} \Delta t f_{n-1} + 6.9669459334684869\,\text{e-01} Y_3 \\
&+ 6.0646225972844570\,\text{e-01} \Delta t F_3, \\
y_{n+1} &= 1.4527906634020011\,\text{e-03} y_{n-6} + 1.8351723030834131\,\text{e-02} y_{n-3} \\
&+ 4.7521482658094340\,\text{e-02} y_{n-2} + 6.8090256545031591\,\text{e-02} y_{n-1} \\
&+ 4.2153263269698360\,\text{e-01} y_n + 1.1542564365902303\,\text{e-03} \Delta t f_{n-6} \\
&+ 1.5974901377839960\,\text{e-02} \Delta t f_{n-3} + 4.1366742377066279\,\text{e-02} \Delta t f_{n-2} \\
&+ 5.9271553481442418\,\text{e-02} \Delta t f_{n-1} + 3.6693787409287626\,\text{e-01} \Delta t f_n \\
&+ 3.2143791984829299\,\text{e-02} Y_2 + 4.1090732242082512\,\text{e-01} Y_4 \\
&+ 2.7980690748119430\,\text{e-02} \Delta t F_2 + 3.5768869986082236\,\text{e-01} \Delta t F_4.
\end{aligned}
$$

HB35. Here $c = 0.868$, $c_{\text{eff}} = 0.174$, and
$\sigma = [0, 0.26527088609801541, 0.60496980053937621, 0.64786880192300389, 0.80986872597453752]^T$.

$$
\begin{aligned}
Y_2 &= 1.7816691007586719\,\text{e-01} y_{n-2} + 3.6109332710455727\,\text{e-01} y_{n-1} \\
&+ 4.6073976281957557\,\text{e-01} y_n + 5.0001145566899656\,\text{e-02} \Delta t f_{n-2} \\
&+ 4.0169387106460058\,\text{e-01} \Delta t f_{n-1} + 5.3100301672280681\,\text{e-01} \Delta t f_n, \\
Y_3 &= 1.0492338206643385\,\text{e-01} y_{n-2} + 3.4085756889984442\,\text{e-01} y_{n-1} \\
&+ 1.6301758867815281\,\text{e-02} \Delta t f_{n-2} + 3.5361621310382257\,\text{e-01} \Delta t f_{n-1} \\
&+ 5.5421904903372177\,\text{e-01} Y_2 + 6.3873798337087584\,\text{e-01} \Delta t F_2, \\
Y_4 &= 3.0725884168559071\,\text{e-02} y_{n-2} + 7.6120687155217670\,\text{e-01} y_n \\
&+ 1.7365123290896886\,\text{e-02} \Delta t f_{n-2} + 3.2628334993259889\,\text{e-01} \Delta t f_n \\
&+ 2.0806724427926448\,\text{e-01} Y_3 + 2.3979769776622345\,\text{e-01} \Delta t F_3, \\
Y_5 &= 1.8056847429213218\,\text{e-02} y_{n-2} + 2.9320650052203379\,\text{e-01} y_{n-1} \\
&+ 3.8292487746719406\,\text{e-01} y_n + 1.4729336401339038\,\text{e-01} \Delta t f_{n-1} \\
&+ 4.4132128702969387\,\text{e-01} \Delta t f_n + 3.0581177458155906\,\text{e-01} Y_4 \\
&+ 3.5244836229981102\,\text{e-01} \Delta t F_4, \\
y_{n+1} &= 2.6873475470074591\,\text{e-02} y_{n-2} + 2.0407159436352018\,\text{e-01} y_{n-1} \\
&+ 2.9931219518195756\,\text{e-01} y_n + 6.6394379781098021\,\text{e-03} \Delta t f_{n-2} \\
&+ 1.1111374386466548\,\text{e-01} \Delta t f_{n-1} + 3.4495759083373073\,\text{e-01} \Delta t f_n \\
&+ 1.1259265017660104\,\text{e-01} Y_2 + 6.9878484228816845\,\text{e-02} \Delta t F_2 \\
&+ 4.0509519685067606\,\text{e-03} Y_3 + 3.5309913283933986\,\text{e-01} Y_5 \\
&+ 4.0694708785818351\,\text{e-01} \Delta t F_5.
\end{aligned}
$$

HB45. Here $c = 1.197$, $c_{\text{eff}} = 0.239$, and
$\sigma = [0, 0.25917166547014575, 0.55010819021145041, 0.69734554315357311, 0.80216988615377616]^T$.

$$
\begin{aligned}
Y_2 &= 1.8733618444244395\,\text{e-02} y_{n-3} + 4.0138780374735444\,\text{e-01} y_{n-1} \\
&+ 5.7987857780840113\,\text{e-01} y_n + 9.4651902096767951\,\text{e-03} \Delta t f_{n-3} \\
&+ 2.2301463815246203\,\text{e-01} \Delta t f_{n-1} + 4.8428049618809460\,\text{e-01} \Delta t f_n, \\
Y_3 &= 3.5927004169645109\,\text{e-02} y_{n-3} + 3.3069854697071399\,\text{e-01} y_{n-1} \\
&+ 1.9297952203785461\,\text{e-02} \Delta t f_{n-3} + 2.7617998412863431\,\text{e-01} \Delta t f_{n-1} \\
&+ 6.3337444885964089\,\text{e-01} Y_2 + 5.2895710258149131\,\text{e-01} \Delta t F_2, \\
Y_4 &= 4.9620900450512116\,\text{e-03} y_{n-3} + 2.7450604997894808\,\text{e-02} y_{n-2} \\
&+ 4.8177234462288970\,\text{e-02} y_{n-1} + 5.4623018810795865\,\text{e-01} y_n \\
&+ 2.2925131428870125\,\text{e-02} \Delta t f_{n-2} + 4.0234793805534269\,\text{e-02} \Delta t f_{n-1} \\
&+ 2.3520312486738987\,\text{e-01} \Delta t f_n + 3.7317988238680644\,\text{e-01} Y_3 \\
&+ 3.1165789792188309\,\text{e-01} \Delta t F_3, \\
Y_5 &= 1.0956495799350385\,\text{e-02} y_{n-3} + 6.6961072534648405\,\text{e-02} y_{n-2} \\
&+ 9.5567193825941879\,\text{e-02} y_{n-1} + 4.8444991951332234\,\text{e-01} y_n \\
&+ 5.5921951031412481\,\text{e-02} \Delta t f_{n-2} + 7.9812101733861315\,\text{e-02} \Delta t f_{n-1} \\
&+ 4.0458409118488264\,\text{e-01} \Delta t f_n + 3.4206531832673703\,\text{e-01} Y_4 \\
&+ 2.8567284329435133\,\text{e-01} \Delta t F_4, \\
y_{n+1} &= 2.2308657456135885\,\text{e-03} y_{n-3} + 1.1153501456345802\,\text{e-01} y_{n-1} \\
&+ 2.6645380153916320\,\text{e-01} y_n + 1.1413620383725557\,\text{e-03} \Delta t f_{n-3} \\
&+ 4.3833314566619347\,\text{e-02} \Delta t f_{n-1} + 2.2252654979647457\,\text{e-01} \Delta t f_n \\
&+ 1.4935602319405525\,\text{e-01} Y_2 + 4.3533635910073039\,\text{e-02} \Delta t F_2 \\
&+ 1.6068914632427829\,\text{e-03} Y_3 + 4.6881740349446721\,\text{e-01} Y_5 \\
&+ 3.9152873286678164\,\text{e-01} \Delta t F_5.
\end{aligned}
$$

HB55. Here $c = 1.410$, $c_{\text{eff}} = 0.282$, and
$\sigma = [0, 0.24479844454438968, 0.52399756081633841, 0.70918789516959513, 0.78535343093546328]^T$.

$$
\begin{aligned}
Y_2 &= 1.4758282323084947\,\text{e-03} y_{n-4} + 1.8241353278840060\,\text{e-02} y_{n-3} \\
&+ 3.5389085596036979\,\text{e-01} y_{n-1} + 6.2639196252848173\,\text{e-01} y_n \\
&+ 1.2939180444763889\,\text{e-02} \Delta t f_{n-3} + 2.0205746303346508\,\text{e-01} \Delta t f_{n-1} \\
&+ 4.4432002979228463\,\text{e-01} \Delta t f_n, \\
Y_3 &= 5.6462353437033699\,\text{e-03} y_{n-4} + 1.1334390031996901\,\text{e-02} y_{n-3} \\
&+ 2.9586763924562143\,\text{e-01} y_{n-1} + 4.0050569071828739\,\text{e-03} \Delta t f_{n-4} \\
&+ 6.9472187302864700\,\text{e-03} \Delta t f_{n-3} + 2.0986846279690249\,\text{e-01} \Delta t f_{n-1} \\
&+ 6.8715173537867824\,\text{e-01} Y_2 + 4.8741889711171360\,\text{e-01} \Delta t F_2, \\
Y_4 &= 6.4702138688064320\,\text{e-05} y_{n-4} + 1.2566485028187580\,\text{e-03} y_{n-1} \\
&+ 5.0510013064449000\,\text{e-01} y_n + 8.9138132928309743\,\text{e-04} \Delta t f_{n-1} \\
&+ 1.0106686220114255\,\text{e-01} \Delta t f_n + 4.9357851871400321\,\text{e-01} Y_3 \\
&+ 3.5011116881926130\,\text{e-01} \Delta t F_3, \\
Y_5 &= 1.6653823505701616\,\text{e-03} y_{n-4} + 6.1661777077330622\,\text{e-02} y_{n-2} \\
&+ 4.5388215525518444\,\text{e-02} y_{n-1} + 5.3513673278857754\,\text{e-01} y_n \\
&+ 4.3738688021238786\,\text{e-02} \Delta t f_{n-2} + 3.2195325739991461\,\text{e-02} \Delta t f_{n-1} \\
&+ 3.7958975095366942 \Delta t f_n + 3.5614789225800331\,\text{e-01} Y_4 \\
&+ 2.5262719122348276\,\text{e-01} \Delta t F_4, \\
y_{n+1} &= 1.5966898078594674\,\text{e-03} y_{n-4} + 1.3580663123776912\,\text{e-02} y_{n-2} \\
&+ 2.7580648991023610\,\text{e-02} y_{n-1} + 3.9205520105793118\,\text{e-01} y_n \\
&+ 1.0112945480867850\,\text{e-03} \Delta t f_{n-4} + 9.6332025388674511\,\text{e-03} \Delta t f_{n-2} \\
&+ 1.9563844229283096\,\text{e-02} \Delta t f_{n-1} + 1.8614377190600623\,\text{e-01} \Delta t f_n \\
&+ 5.6518679701940866\,\text{e-01} Y_5 + 4.0090523108915388\,\text{e-01} \Delta t F_5.
\end{aligned}
$$

HB65. Here $c = 1.463$, $c_{\text{eff}} = 0.293$, and
$\sigma = [0, 0.23246377384708192, 0.50781889103199529, 0.69900628524241393, 0.79267611638163260]^T$.

$$\begin{aligned}
Y_2 &= 5.4783430788371654\,\text{e-04}y_{n-5} + 2.7369198960966935\,\text{e-02}y_{n-3} \\
&+ 3.4867751470068636\,\text{e-01}y_{n-1} + 6.2340545203046283\,\text{e-01}y_n \\
&+ 1.8705680226652550\,\text{e-02}\Delta t f_{n-3} + 2.2121133776766960\,\text{e-01}\Delta t f_{n-1} \\
&+ 4.2607103897576548\,\text{e-01}\Delta t f_n, \\
Y_3 &= 5.9026394535461740\,\text{e-03}y_{n-4} + 8.6222222130342870\,\text{e-03}y_{n-3} \\
&+ 2.8824255361854179\,\text{e-01}y_{n-1} + 4.0342023260792233\,\text{e-03}\Delta t f_{n-4} \\
&+ 5.8929211552803742\,\text{e-03}\Delta t f_{n-3} + 1.9700149220266783\,\text{e-01}\Delta t f_{n-1} \\
&+ 6.9723258471487770\,\text{e-01}Y_2 + 4.7652873552782138\,\text{e-01}\Delta t F_2, \\
Y_4 &= 1.1155043080192375\,\text{e-03}y_{n-4} + 9.6574743746123855\,\text{e-04}y_{n-3} \\
&+ 4.4020233713525536\,\text{e-02}y_{n-1} + 4.4101169641102805\,\text{e-01}y_n \\
&+ 7.6239961962423428\,\text{e-04}\Delta t f_{n-4} + 6.6004718554688903\,\text{e-04}\Delta t f_{n-3} \\
&+ 3.0085952333572669\,\text{e-02}\Delta t f_{n-1} + 1.0788749796338853\,\text{e-01}\Delta t f_n \\
&+ 5.1288681812996595\,\text{e-01}Y_3 + 3.5053626619057981\,\text{e-01}\Delta t F_3, \\
Y_5 &= 3.6331156100948995\,\text{e-04}y_{n-5} + 4.6629159718212201\,\text{e-02}y_{n-2} \\
&+ 9.3017395878286502\,\text{e-03}y_{n-1} + 5.5340849082700760\,\text{e-01}y_n \\
&+ 3.1869041990243555\,\text{e-02}\Delta t f_{n-2} + 6.3573423003597807\,\text{e-03}\Delta t f_{n-1} \\
&+ 3.1925452660602294\,\text{e-01}\Delta t f_n + 3.9029729830594206\,\text{e-01}Y_4 \\
&+ 2.6675155768532000\,\text{e-01}\Delta t F_4, \\
y_{n+1} &= 6.4050839320003712\,\text{e-04}y_{n-5} + 5.0178296130050076\,\text{e-03}y_{n-3} \\
&+ 1.5366630320738119\,\text{e-02}y_{n-2} + 7.0874655850485691\,\text{e-02}y_{n-1} \\
&+ 3.4127150555948699\,\text{e-01}y_n + 4.3776016984545496\,\text{e-04}\Delta t f_{n-5} \\
&+ 3.4294725361367802\,\text{e-03}\Delta t f_{n-3} + 1.0502436456063365\,\text{e-02}\Delta t f_{n-2} \\
&+ 4.8439804555624320\,\text{e-02}\Delta t f_{n-1} + 2.2033940778576508\,\text{e-01}\Delta t f_n \\
&+ 5.6682887026308426\,\text{e-01}Y_5 + 3.8740335826041222\,\text{e-01}\Delta t F_5.
\end{aligned}$$

HB75. Here $c = 1.481$, $c_{\text{eff}} = 0.296$, and
$\sigma = [0, 0.22374103119231895, 0.49727709563768979, 0.67144968190576249, 0.80283471701709541]^T$.

$$\begin{aligned}
Y_2 &= 2.1921061962799072\,\text{e-04}y_{n-6} + 6.8960132210314936\,\text{e-04}y_{n-4} \\
&+ 3.1882256073990631\,\text{e-02}y_{n-3} + 3.5136798162144256\,\text{e-01}y_{n-1} \\
&+ 6.1584095036283570\,\text{e-01}y_n + 4.6546531588335515\,\text{e-04}\Delta t f_{n-4} \\
&+ 2.1519802701791095\,\text{e-02}\Delta t f_{n-3} + 2.3716545098540029\,\text{e-01}\Delta t f_{n-1} \\
&+ 4.1567873103883923\,\text{e-01}\Delta t f_n, \\
Y_3 &= 5.4550790706818335\,\text{e-03}y_{n-4} + 8.7424655724822262\,\text{e-03}y_{n-3} \\
&+ 2.8618560022708417\,\text{e-01}y_{n-1} + 3.6820551547953636\,\text{e-03}\Delta t f_{n-4} \\
&+ 5.9009667888747439\,\text{e-03}\Delta t f_{n-3} + 1.9316881586697723\,\text{e-01}\Delta t f_{n-1} \\
&+ 6.9961685512975180\,\text{e-01}Y_2 + 4.7222557444804270\,\text{e-01}\Delta t F_2, \\
Y_4 &= 3.2103306707334349\,\text{e-11}y_{n-6} + 1.9753684143828873\,\text{e-04}y_{n-4} \\
&+ 3.3807025207674979\,\text{e-03}y_{n-1} + 5.0661048515384699\,\text{e-01}y_n \\
&+ 1.3333290605977094\,\text{e-04}\Delta t f_{n-4} + 2.2818978390841127\,\text{e-03}\Delta t f_{n-1} \\
&+ 9.9021825572859906\,\text{e-02}\Delta t f_n + 4.8981127545184378\,\text{e-01}Y_3 \\
&+ 3.3061154719961389\,\text{e-01}\Delta t F_3, \\
Y_5 &= 1.1329844228955921\,\text{e-04}y_{n-6} + 4.7216781448026698\,\text{e-04}y_{n-3} \\
&+ 2.7972381455310742\,\text{e-02}y_{n-2} + 5.2775994407848248\,\text{e-01}y_n \\
&+ 3.6354713306523248\,\text{e-05}\Delta t f_{n-6} + 3.1870260956973120\,\text{e-04}\Delta t f_{n-3} \\
&+ 1.8880725649418614\,\text{e-02}\Delta t f_{n-2} + 2.4425424472070059\,\text{e-01}\Delta t f_n \\
&+ 4.4368220820943699\,\text{e-01}Y_4 + 2.9947546876242703\,\text{e-01}\Delta t F_4, \\
y_{n+1} &= 5.4496586863497548\,\text{e-04}y_{n-5} + 1.1080171578611788\,\text{e-03}y_{n-4} \\
&+ 2.8478009220581146\,\text{e-02}y_{n-2} + 5.7441893113528306\,\text{e-02}y_{n-1} \\
&+ 3.5617638481313241\,\text{e-01}y_n + 3.6783965178054497\,\text{e-04}\Delta t f_{n-5} \\
&+ 7.4788655395136721\,\text{e-04}\Delta t f_{n-4} + 1.9222012969986973\,\text{e-02}\Delta t f_{n-2} \\
&+ 3.8771980369009612\,\text{e-02}\Delta t f_{n-1} + 2.4041101452882352\,\text{e-01}\Delta t f_n \\
&+ 5.5625072982626200\,\text{e-01}Y_5 + 3.7545667818513806\,\text{e-01}\Delta t F_5.
\end{aligned}$$

HB36. Here $c = 1.395$, $c_{\text{eff}} = 0.232$, and $\sigma = [0, 0.22599458000683792, 0.48158277187432164, 0.68829197344352122, 0.62314864046524110, 0.83924050761030644]^T$.

$$\begin{aligned}
Y_2 &= 1.0841309922552752\,\text{e-02}y_{n-3} + 2.9380436878075455\,\text{e-01}y_{n-1} \\
&+ 6.9535432129669272\,\text{e-01}y_n + 5.1444690209902231\,\text{e-03}\Delta t f_{n-3} \\
&+ 1.5336383758578842\,\text{e-01}\Delta t f_{n-1} + 3.9972874293264599\,\text{e-01}\Delta t f_n, \\
Y_3 &= 5.2888158945964478\,\text{e-03}y_{n-3} + 2.3109985410026951\,\text{e-01}y_{n-1} \\
&+ 8.1774718095017906\,\text{e-04}\Delta t f_{n-3} + 1.3284918399464901\,\text{e-01}\Delta t f_{n-1} \\
&+ 7.6361133000513404\,\text{e-01}Y_2 + 4.3896670759573791\,\text{e-01}\Delta t F_2, \\
Y_4 &= 3.6566210966630479\,\text{e-04}y_{n-3} + 4.6969649343724151\,\text{e-01}y_n \\
&+ 2.1020313091956204\,\text{e-04}\Delta t f_{n-3} + 9.4872475991182009\,\text{e-02}\Delta t f_n \\
&+ 5.2993784445309222\,\text{e-01}Y_3 + 3.0463805560401014\,\text{e-01}\Delta t F_3 \\
Y_5 &= 1.8768070135261744\,\text{e-04}y_{n-3} + 3.6352874982539300\,\text{e-02}y_{n-2} \\
&+ 6.8355417080780034\,\text{e-01}y_n + 1.0788941483026889\,\text{e-04}\Delta t f_{n-3} \\
&+ 2.0897675578775733\,\text{e-02}\Delta t f_{n-2} + 3.3820253121259541\,\text{e-01}\Delta t f_n \\
&+ 2.7990527350830763\,\text{e-01}Y_4 + 1.6090528194467757\,\text{e-01}\Delta t F_4 \\
Y_6 &= 2.4405614580881880\,\text{e-03}y_{n-3} + 2.0248646751064373\,\text{e-03}y_{n-2} \\
&+ 1.1277016933764120\,\text{e-01}y_{n-1} + 2.5589293062236024\,\text{e-01}y_n \\
&+ 1.1640060130490865\,\text{e-03}\Delta t f_{n-2} + 6.4826631041246308\,\text{e-02}\Delta t f_{n-1} \\
&+ 1.4710163775538382\,\text{e-01}\Delta t f_n + 6.2687147390680387\,\text{e-01}Y_5 \\
&+ 3.6036095350327912\,\text{e-01}\Delta t F_5 \\
y_{n+1} &= 3.2655021954324220\,\text{e-04}y_{n-3} + 3.7407656160088655\,\text{e-03}y_{n-2} \\
&+ 5.5711981665515067\,\text{e-02}y_{n-1} + 2.1278106986814990\,\text{e-01}y_n \\
&+ 1.8771941838081294\,\text{e-04}\Delta t f_{n-3} + 2.1504023078543280\,\text{e-03}\Delta t f_{n-2} \\
&+ 1.8637402963501817\,\text{e-02}\Delta t f_{n-1} + 1.2231851729870541\,\text{e-01}\Delta t f_n \\
&+ 1.6098234510432394\,\text{e-01}Y_2 + 9.2541699205813038\,\text{e-02}\Delta t F_2 \\
&+ 4.0533039944613838\,\text{e-02}Y_3 + 2.3300669325079338\,\text{e-02}\Delta t F_3 \\
&+ 5.2592424758184542\,\text{e-01}Y_6 + 3.0233081455747418\,\text{e-01}\Delta t F_6.
\end{aligned}$$

HB56. Here $c = 1.804$, $c_{\text{eff}} = 0.301$, and
$\sigma = [0, 0.22645788718120133, 0.49233915315493260, 0.65120998281005327, 0.64798790448485055, 0.83882922436916785]^T$.

$$\begin{aligned}
Y_2 &= 1.3644139802438684\,\text{e-03}y_{n-4} + 9.4825639133235707\,\text{e-03}y_{n-3} \\
&+ 6.8544116302425486\,\text{e-03}y_{n-2} + 2.8034611580634328\,\text{e-01}y_{n-1} \\
&+ 7.0195249466984677\,\text{e-01}y_n + 7.5645590738268804\,\text{e-04}\Delta t f_{n-4} \\
&+ 5.2573057687999849\,\text{e-03}\Delta t f_{n-3} + 3.8002103792595613\,\text{e-03}\Delta t f_{n-2} \\
&+ 1.5542898158782706\,\text{e-01}\Delta t f_{n-1} + 3.8917522026570661\,\text{e-01}\Delta t f_n, \\
Y_3 &= 2.6218830337355159\,\text{e-03}y_{n-4} + 5.4407306131174383\,\text{e-04}y_{n-3} \\
&+ 2.2479385861492829\,\text{e-01}y_{n-1} + 1.4536196037665699\,\text{e-03}\Delta t f_{n-4} \\
&+ 3.0164399312552354\,\text{e-04}\Delta t f_{n-3} + 1.2462980059924102\,\text{e-01}\Delta t f_{n-1} \\
&+ 7.7204018529002461\,\text{e-01}Y_2 + 4.2803310971284286\,\text{e-01}\Delta t F_2, \\
Y_4 &= 3.1165475938829245\,\text{e-04}y_{n-3} + 4.6398517074649165\,\text{e-01}y_n \\
&+ 1.7278706259009910\,\text{e-04}\Delta t f_{n-3} + 9.1220936992512242\,\text{e-02}\Delta t f_n \\
&+ 5.3570317449411997\,\text{e-01}Y_3 + 2.9700357576027159\,\text{e-01}\Delta t F_3 \\
Y_5 &= 3.3648602862423341\,\text{e-02}y_{n-2} + 1.2156286786952630\,\text{e-02}y_{n-1} \\
&+ 6.3902216041310722\,\text{e-01}y_n + 1.8655396953573083\,\text{e-02}\Delta t f_{n-2} \\
&+ 6.7396663219361150\,\text{e-03}\Delta t f_{n-1} + 3.2206495104027316\,\text{e-01}\Delta t f_n \\
&+ 3.1517294993751704\,\text{e-01}Y_4 + 1.7473761136986321\,\text{e-01}\Delta t F_4 \\
Y_6 &= 5.6371309272195845\,\text{e-04}y_{n-4} + 1.0046734664416114\,\text{e-02}y_{n-2} \\
&+ 8.3275561963346595\,\text{e-02}y_{n-1} + 3.0371467972901972\,\text{e-01}y_n \\
&+ 5.5700922863935812\,\text{e-03}\Delta t f_{n-2} + 4.6169484994962436\,\text{e-02}\Delta t f_{n-1} \\
&+ 1.6838493812471231\,\text{e-01}\Delta t f_n + 6.0239931055049534\,\text{e-01}Y_5 \\
&+ 3.3398112571943184\,\text{e-01}\Delta t F_5 \\
y_{n+1} &= 1.9217139490270655\,\text{e-06}y_{n-4} + 2.7038477553249367\,\text{e-04}y_{n-3} \\
&+ 1.1370562101640887\,\text{e-02}y_{n-2} + 4.6847192492490974\,\text{e-03}y_{n-1} \\
&+ 2.1595496982660728\,\text{e-01}y_n + 1.0654331383873979\,\text{e-06}\Delta t f_{n-4} \\
&+ 1.4990623350354576\,\text{e-04}\Delta t f_{n-3} + 6.3040462767102769\,\text{e-03}\Delta t f_{n-2} \\
&+ 2.5972934914451837\,\text{e-03}\Delta t f_{n-1} + 1.1972936001783474\,\text{e-01}\Delta t f_n \\
&+ 3.4614524260998362\,\text{e-02}Y_2 + 1.5726396362708908\,\text{e-03}\Delta t F_2 \\
&+ 1.8281085202699393\,\text{e-01}Y_3 + 3.3353712131245231\,\text{e-02}\Delta t F_3 \\
&+ 5.5029206604502878\,\text{e-01}Y_6 + 3.0509192237328558\,\text{e-01}\Delta t F_6.
\end{aligned}$$

HB66. Here $c = 1.828$, $c_{\text{eff}} = 0.305$, and
$\sigma = [0, 0.22263002041236041, 0.48693561463884488,$
$0.64733846114459725, 0.64342678780115270,$
$0.83094581762751496]^T.$

$$
\begin{aligned}
Y_2 &= 4.8747771145061960\,\text{e-04} y_{n-4} + 1.2199100700146447\,\text{e-02} y_{n-3} \\
&+ 2.8564979587035266\,\text{e-01} y_{n-1} + 7.0166362571805030\,\text{e-01} y_n \\
&+ 2.6664634424189773\,\text{e-04} \Delta t f_{n-4} + 6.5079696081076907\,\text{e-03} \Delta t f_{n-3} \\
&+ 1.5624811558174562\,\text{e-01} \Delta t f_{n-1} + 3.8380429769485963\,\text{e-01} \Delta t f_n, \\
Y_3 &= 2.5424247167602373\,\text{e-03} y_{n-4} + 1.9917765558595535\,\text{e-04} y_{n-3} \\
&+ 2.2190578909801154\,\text{e-01} y_{n-1} + 1.3906856463588686\,\text{e-03} \Delta t f_{n-4} \\
&+ 1.0894855799375904\,\text{e-04} \Delta t f_{n-3} + 1.2138066221122475\,\text{e-01} \Delta t f_{n-1} \\
&+ 7.7535260852964216\,\text{e-01} Y_2 + 4.2411157209134681\,\text{e-01} \Delta t F_2, \\
Y_4 &= 5.9902897465687402\,\text{e-05} y_{n-4} + 4.1705923196541303\,\text{e-05} y_{n-3} \\
&+ 4.5675852704024805\,\text{e-01} y_n + 3.2766397813732522\,\text{e-05} \Delta t f_{n-4} \\
&+ 2.2812800857110321\,\text{e-05} \Delta t f_{n-3} + 8.6080378579914768\,\text{e-02} \Delta t f_n \\
&+ 5.4313986413909032\,\text{e-01} Y_3 + 2.9709308914603760\,\text{e-01} \Delta t F_3 \\
Y_5 &= 2.6650291743336280\,\text{e-11} y_{n-5} + 3.3168109756834617\,\text{e-02} y_{n-2} \\
&+ 1.7971839811495915\,\text{e-02} y_{n-1} + 6.3340798846857871\,\text{e-01} y_n \\
&+ 1.8142686330733744\,\text{e-02} \Delta t f_{n-2} + 9.8304502389970766\,\text{e-03} \Delta t f_{n-1} \\
&+ 3.2300774556803685\,\text{e-01} \Delta t f_n + 3.1545206193644038\,\text{e-01} Y_4 \\
&+ 1.7254971278297584\,\text{e-01} \Delta t F_4 \\
Y_6 &= 3.8045947857330136\,\text{e-04} y_{n-5} + 1.0167623736064991\,\text{e-02} y_{n-2} \\
&+ 8.0697598781129393\,\text{e-02} y_{n-1} + 3.0745759552577484\,\text{e-01} y_n \\
&+ 2.0678234811777690\,\text{e-04} \Delta t f_{n-5} + 5.5616075056655473\,\text{e-03} \Delta t f_{n-2} \\
&+ 4.4140930341311946\,\text{e-02} \Delta t f_{n-1} + 1.6817680466328971\,\text{e-01} \Delta t f_n \\
&+ 6.0129672247845734\,\text{e-01} Y_5 + 3.2890441775558077\,\text{e-01} \Delta t F_5 \\
y_{n+1} &= 3.6499739071497420\,\text{e-04} y_{n-4} + 1.3921027337489261\,\text{e-02} y_{n-2} \\
&+ 2.9948862055808489\,\text{e-02} y_{n-1} + 3.0786074862208745\,\text{e-01} y_n \\
&+ 1.9965060474733641\,\text{e-04} \Delta t f_{n-4} + 7.6146887548692085\,\text{e-03} \Delta t f_{n-2} \\
&+ 1.6381784015562784\,\text{e-02} \Delta t f_{n-1} + 1.6839732612873615\,\text{e-01} \Delta t f_n \\
&+ 4.8914399830795791\,\text{e-08} Y_2 + 7.5911491743955284\,\text{e-02} Y_3 \\
&+ 4.1522968710174052\,\text{e-02} \Delta t F_3 + 5.7199282393554496\,\text{e-01} Y_6 \\
&+ 3.1287542353705583\,\text{e-01} \Delta t F_6.
\end{aligned}
$$

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Truong Nguen-Ba, Huong Nguen-Thu, Tjeri Giordanos, Remi Vajenkurs. Ermita-Birkhofa k-soļu 7. kārtas laika diskretizācija no 6. līdz 10. etapam, kas saglabā stingru stabilitāti
Nestacionāru parciālo vienādojumu diskretizācija pēc telpiskās koordinātas ar taišņu metodi noved pie parastu diferenciālvienādojumu sistēmas. Rakstā aplūkota daudzsoļu metožu saime iegūtās parasto diferenciālvienādojumu sistēmas skaitliskai integrēšanai. Nelineāras stabilitātes īpašība, kas piemīt laika diskretizācijas metodēm, kuri saglabā stingro stabilitāti (SSS), ir īpaši piemērota hiperbolisko nezūdamības likumu integrēšanai. Rakstā ir konstruēta k soļu atklāto Ermita-Birkhoffa metožu saime, kuri saglabā stingro stabilitāti, ar kārtu 7 ar nenegatīviem koeficientiem. Metodes ir konstruētas uz lineāras k soļu ceturtās kārtas metodes kombinēšanu ar ceturtās kārtas no sestā līdz desmitam etapam Runges-Kutta metodēm. Piedāvātām metodēm ir lielākā efektivitāte attiecībā pret Kuranta-Fridriksa-Levi (KFL) koeficientiem. Parādīts, ka sestās kārtas Ermita-Birkhofa metodei ir lielākais efektīvais SSS koeficients starp zināmajām septītās kārtas Ermita-Birkhofa metodēm. Visām jaunajām Ermita-Birkhofa metodēm ir lielāks efektīvais SSS koeficients un lielāki maksimālās efektivitātes KFL skaitļi nekā Huanga septītās kārtas hibrīda metodei vai Burgera vienādojumiem, neatkarīgi no soļu skaita k. Rakstā aplūkoti divi piemēri, kas ilustrē piedāvāto metožu efektivitāti. Abos piemēros izmantots Burgera vienādojums ar dažādiem sākuma nosacījumiem.

Труонг Нгуен-Ба, Хуонг Нгуен-Ху, Тьери Джордано, Реми Вайенкур. K-шаговые, от 6 до 10 этапов, дискретизации по времени Эрмита-Биркхофа 7 порядка, сохраняющие сильную устойчивость
Дискретизация нестационарных уравнений в частных производных по пространственной координате с помощью метода прямых приводит к системе обыкновенных дифференциальных уравнений. В статье рассматривается семейство многошаговых методов для численного интегрирования полученной системы обыкновенных дифференциальных уравнений. Нелинейное свойство устойчивости методов дискретизации по времени, сохраняющих сильную устойчивость, делает их особенно привлекательными для интегрирования гиперболических законов сохранения. В статье приводится набор к-шаговых явных, сохраняющих сильную устойчивость (ССУ), методов Эрмита-Биркхофа (ССУ ЭБ) седьмого порядка с неотрицательными коэффициентами, построенных по принципу комбинирования к-шаговых методов четвертого порядка с шести до десяти-этапными методами Рунге-Кутты четвертого порядка. Показано, что шести-шаговый метод Эрмита-Биркхофа шестого порядка имеет наибольший эффективный коэффициент ССУ по сравнению с известными методами Эрмита-Биркхофа седьмого порядка. Предлагаемые методы имеют, вообще говоря, более эффективные коэффициенты Куранта-Фридрикса-Леви (КФЛ). У всех новых методов Эрмита-Биркхофа эффективный коэффициент ССУ и числа максимальной эффективности КФЛ больше, чем у гибридного метода Хуанга седьмого порядка, также они больше по сравнению с уравнениями Бюргера, вне зависимости от числа шагов к. В статье приведены два примера, иллюстрирующие эффективность предложенных методов. Оба примера используют уравнение Бюргера с различными начальными условиями.