RIGA TECHNICAL UNIVERSITY
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study program "Mechanical Engineering"

# GLOBAL ANALYSIS OF DYNAMICS OF THE PENDULUM SYSTEMS, NEW BIFURCATION GROUPS AND RARE ATTRACTORS

**Summary of Doctoral Thesis**

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**DOCTORAL THESIS**
**SUBMITTED FOR THE DEGREE OF DOCTOR OF ENGINEERING SCIENCES (Engineering Techniques, Mechanics and Mechanical Engineering) TO RIGA TECHNICAL UNIVERSITY**

The doctoral thesis submitted for the degree of Doctor of Engineering Sciences (Engineering Techniques, Mechanics and Mechanical Engineering) will be defended at the public session at the Faculty of Transport and Machinery, Riga Technical University, at 29 of November, 2013 at 6 Ezermalas Street, room 302.

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CONFIRMATION

I confirm that I have independently worked out this thesis for defense at Riga Technical University for being conferred the degree of Doctor of Science in Engineering. The thesis hasn't been presented to any other university for obtaining the scientific degree.

Alexey Klokov …………………………….

Date: ………………………

The thesis has been written in English. The thesis contains the introduction, 6 chapters, conclusions, reference list, 3 appendices, 113 figures and 8 tables; the total number of pages is 191. The reference list contains 152 items.

# CONTENTS

# GENERAL DESCRIPTION OF THE THESIS

## General description

The doctoral thesis present the results of the global bifurcation analysis, new bifurcation groups, rare attractors and chaos for pendulum driven systems and researches conducted on “novelty bifurcation theory”, developed in Institute of Mechanics of Riga Technical University (RTU). The method of complete bifurcation groups is developed (1993-2013) in Institute of Mechanics of Riga Technical University in the research group of Prof. M. Zakrzhevsky. This group is working in such scientific direction “Nonlinear dynamics, chaos, catastrophes and control”. The method of complete bifurcation groups and complex of algorithms and software proposed on its basis allows finding for the archetypical and widely used classical nonlinear dynamic models qualitatively new previously unknown rare regular and chaotic attractors, new bifurcation groups and studying the interaction of different bifurcation groups.

## The topicality of the thesis

Design of machines, mechanisms and appliances requires modern knowledges in the region of nonlinear dynamics and nonlinear oscillation theory. For today there are worked out a lot of typical nonlinear dynamic models, which are being broadly used for analysis and synthesis of various technical systems and technological processes. They are used, for example, for calculations of vehicles (air, auto and railway) vibrations, of vibro-insulation appliances, oscillation of solid systems, various mechanisms and appliances with impact interaction, gears, electromechanical and electric appliances and many other systems.

However, in the present time, there is a situation when many important regimes, even in typical driven damped pendulum systems, remains unnoticed while modeling by traditional analytic or numeral methods, despite modern opportunities of application of high-speed computers. This conclusion is right even for simple pendulum models, which are, for example, pendulum with external harmonic excitation, pendulum with the vibrating point of suspension in the vertical direction etc.

In the present thesis there are analyzed the reasons for imperfection of traditional methods of nonlinear dynamics and laid out the basis of new alternative approach, which got the name “complete bifurcation groups method" and presented new qualitative research results, which are got by using the method mentioned above.

Basic statements of the method of complete bifurcation groups for the analysis of the driven damped pendulum systems presents scientific and practical interest, because of existing methods don't allow finding systematically all existing regimes in a system, and that, in its turn, keeps back from practical application of nonlinear effects, existing in such systems. This circumstance defines the topicality of the present doctoral thesis.

## The main aim of this thesis

The main aim of the present doctoral thesis is to show, that the method of complete bifurcation groups allows conducting the global bifurcation analysis of the pendulum systems with one or several degrees of freedom and finding new bifurcation groups and unknown before forced regular and chaotic oscillations, and showing the possibility of using new (found regimes) results in real pendulum-like systems.

## The main tasks of the research

It is well known that the simple nonlinear dynamical systems have quite complex behavior. To date, almost all archetypical models of nonlinear dynamical systems are understudied and many important stable regimes remain unknown. This is also applied fully to the driven damped pendulum systems with one or several degrees of freedom.

The main task of the doctoral thesis is a theoretical and practical study of the nonlinear dynamics of the pendulum system based on the method of complete bifurcation groups, which allow conducting the global bifurcation analysis and finding new bifurcation groups, rare periodic and chaotic attractors both oscillating and rotating. The results can provide guidance on the use of complex regular and chaotic regimes on real objects in vibroengineering, robotics, space dynamics, medicine etc.

In the present doctoral thesis it is planned to show the typical bifurcation groups and their topology change with varying of parameters in the driven damped pendulum systems with one or several degrees of freedom. The author hope that the materials of this thesis will be useful both for understanding the qualitative behavior of strongly nonlinear dynamical systems and for the practical use of new understudied phenomena in pendulum systems, in particular multiplicity, the coexistence of periodic and chaotic attractors, different chaotic attractors, rare attractors, as well as chaotic rotating regimes.

The materials of the doctoral thesis may be used in the teaching process: in courses of ordinary differential equations, nonlinear oscillations in nature and engineering, nonlinear theory of chaos and control.

## The scientific novelty and main results

Main scientific novelty of the present doctoral thesis was defined by such main results:

1. Apparently, rare regular and chaotic rotational regimes have been found in the pendulum systems with or several degrees of freedom for the first time within the framework of this research. These nonlinear phenomena define a great theoretical and practical interest of studies of nonlinear dynamics in the pendulum systems.
2. It is shown that the use of the method of complete bifurcation groups allows conducting the qualitative global bifurcation analysis of the pendulum systems and finding the new bifurcation groups and previously unknown regular and chaotic oscillating, oscillating-rotating and rotating orbits (regimes, attractors) by the example of the simplest systems with one or more degrees of freedom. The qualitative topology of different bifurcation groups is investigated and new periodic and chaotic orbits are found.
3. The birth of the previously unknown rare attractors has been shown for different harmonically driven damped systems using the method of complete bifurcation groups, and new bifurcation groups with complex protuberances have been obtained. The new types of interaction of different oscillating and rotating orbits have been found as well as rare and chaotic rotational regimes.
4. Apparently, the process of formation of chaotic rotation through the cascade of period-doubling bifurcations for different groups has been found for the first time and studied within the framework of the research.
5. The interaction of different bifurcation groups in the pendulum systems with one or several degrees of freedom is investigated. Two-parameter bifurcation diagrams were constructed. It is shown, that the existence of bifurcation subgroup with unstable periodic infinitium (UPI) always leads to chaotic behavior of the system: chaotic attractor or transient chaos
6. The main qualitative results of topology of bifurcation groups with rare regular and chaotic attractors for the pendulum systems with two degrees of freedom were investigated. The subharmonic, chaotic and rare periodic behaviour were investigated. There are complex protuberances with many rare regular attractors of different types, chaotic transients and chaotic motions.

7. The possibility of using the method of complete bifurcation groups for the global analysis of the six body symmetric driven pendulum systems with several equilibrium positions is demonstrated.

8. The possibility of performance of the experimental investigations in more realistic models of pendulum systems, which were discussed in the present doctoral thesis, is shown. The visualization possibility of the founded regimes by the method of complete bifurcation groups in the pendulum systems is investigated as well.

Thus, in the doctoral thesis it is was shown, that the method of complete bifurcation groups allows conducting the global bifurcation analysis of the pendulum systems with one or several degrees of freedom and finding new nonlinear phenomena, new bifurcation groups, unknown before rare periodic and chaotic attractors both oscillating and rotating. The possibility of usage of new obtained results in real objects is discussed as well.

The author is of the opinion that the method of complete bifurcation groups will be useful for global bifurcation analysis and for search of rare attractors as well as for other pendulum systems.

## Practical significance

The results obtained by the method of complete bifurcation groups in the pendulum systems, from my point of view have a great practical significance. The new types of interaction of different oscillating and rotating orbits have been found as well as rare and chaotic rotating regimes. Also the process of formation of chaotic rotation through the cascade of period-doubling bifurcations for different groups has been found within the framework of the research. Also attention is paid to stable periodic and chaotic oscillations of different type near unstable equilibrium position of the pendulum, which can be widely used in real objects. There complex protuberances with many rare regular attractors of different types, chaotic transients and chaotic motions have importance in the study of driven damped pendulum systems. The possibility of performance of the experimental investigations and visualization of the founded regimes by the method of complete bifurcation groups in more realistic models of pendulum systems is shown.

The doctoral thesis is intended for use in a course of ordinary differential equations, in courses on nonlinear dynamics and chaos and in nonlinear oscillation theory for students of different specialties who have basic knowledge to the extent of the 1st year of training at technical colleges and universities, as well as for those interested in contemporary issues and methods of nonlinear oscillation theory and nonlinear dynamics and chaos. This doctoral thesis obviously does not cover the topic fully and author will be grateful for remarks.

This doctoral thesis can be useful for illustrating the main statements of the bifurcation theory of nonlinear dynamical systems, which is a rather new section of the general theory of nonlinear dynamical systems.

## Proposed statement for defense

The proposed statement of the doctoral thesis is a theoretical and practical study of the nonlinear dynamics of the pendulum systems with one or several degrees of freedom by the method of complete bifurcation groups, which allow conducting the global bifurcation analysis and finding new bifurcation groups, rare periodic and chaotic attractors both oscillating and rotating. The results can provide guidance on the use of complex regular and chaotic regimes on real objects in vibroengineering, robotics, space dynamics, medicine etc.

## The cooperation with enterprises

During the preparation doctoral thesis, cooperation was carried out with famous German company LuK GmbH & Co. OHG. It is well-known in industry and automotive systems. These investigations were devoted to the centrifugal pendulum vibration absorber for neutralizing the irregularity of the engine torque in automotive. It is an effective device for damping vibrations in power train especially in combination with Dual-Mass Flywheel (DMFW), and its qualitative behavior with taking into account non-linear effects is very important. The important unknown regular or chaotic attractors and new bifurcation groups with rare attractors were found and presented in Paragraph 3.8. The method of complete bifurcation groups gives robust stability to the pendulum vibration absorber by changing the system parameters, that allows saving characteristics of the vibration absorber system and excluding other amazing regimes.

## The approbation of the thesis

Results of the thesis are presented and discussed on 27 international conferences, exhibitions and scientific seminars:

1. Клоков А., Закржевский М.В. Редкие аттракторы в динамике космических аппаратов // XVI Симпозиум «Динамика виброударных (сильно нелинейных) систем» (“DYVIS-2009”), Москва, Звенигород, 24-30 май 2009 г.

2. Klokov A., Zakrzhevsky M., Shilvan E. „Nonlinear Dynamics and Rare Attractors in Pendulum Systems”, 8th International conference„VIBROENGINEERING 2009”, Klaipeda, Lithuania, September 16-18, 2009.
3. Klokov A., Zakrzhevsky M. „Bifurcation Analysis and Rare Attractors in Parametrically Excited Pendulum Systems”. RTU 50. starptautiskā zinātniskā konference par godu Rīgas Tehniskās universitātes 147. jubilejai 12. - 16. oktobrī 2009. gadā apakšsekcijā «Inženiertehnika, mehānika un mašīnbūve».
4. Klokov A., Zakrzhevsky M., et al. „Method of Complete Bifurcation Groups and its Application in Nonlinear Dynamics”. RTU 50. starptautiskā zinātniskā konference par godu Rīgas Tehniskās universitātes 147. jubilejai 12. - 16. oktobrī 2009. gadā apakšsekcijā «Inženiertehnika, mehānika un mašīnbūve».
5. Klokov A., Zakrzhevsky M. „Rare Attractors in the Parametrically excited Pendulum System”, 10th conference on Dynamical Systems - Theory and Applications „DSTA-2009”, Łódź, Poland, December 07-10, 2009.
6. Zakrzhevsky M., Klokov A., Yevstignejev V., Shilvan E., Kragis A. „Nonlinear Dynamics and Rare Attractors in Driven Damped Pendulum Systems”, 7th International DAAAM Baltic Conference „INDUSTRIAL ENGINEERING”, 22-24 April 2010, Tallin, Estonia.
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9. M. Zakrzhevsky, Schukin I., Smirnova R., Yevstigņejev V., Frolov V., Klokov A. Rare periodic and rare chaotic attractors and bifurcation groups in typical nonlinear dynamical discrete models. „16th International Conference on Difference Equations and Applications”, Latvia, Riga, 19.-23. July, 2010.
10. Klokovs A., Zakrževskis M. un citi. „Kompleksā bifurkācijas analīze un retie atraktori diskrētās dināmiskās sistēmās”, RTU 51. Starptautiskā Zinātniskā Konference,11.-15. Oktobris, 2010. gadā, Rīga, Latvija.
11. Klokovs A. „Svārstu svārstības un divu parametru bifurkāciju analīze”, RTU 51. Starptautiskā Zinātniskā Konference, 11.-15. Oktobris, 2010. gadā, Rīga, Latvija.

12. Klokov A., Zakrzhevsky M. „Bifurcation Analysis and Rare Attractors in Driven Damped Pendulum Systems”, 9th International conference „VIBROENGINEERING 2010”, 14-15 October 2010, Kaunas, Lithuania.
13. Klokov A., Zakrzhevsky M. „Method of Complete Bifurcation Groups and its Application in Dynamics of the Machines and Mechanisms”, XXII International Innovation Conference of Young Scientists and Students (IICYSS-2010) “Future of Russian Mechanical Engineering”, October 26 - 29, 2010, Moscow, Russia.
14. Клоков А., Закржевский М. «Новые методы повышения надежности машин на основе бифуркационного анализа», XIII Международная специализированная выставка MASHEX/Машиностроение, Международный выставочный комплекс «Крокус Экспо», 26-28 октября, 2010 года, Москва, Россия.
15. Klokov A., Zakrzhevsky M. „Rare Periodic and Rare Chaotic Attractors and Bifurcation Groups in Discrete Model of Von Karman Streets”, FILOSE semi-annual meeting, November 11 - 12, 2010, RTU, Riga, Latvia.
16. Klokov A., Zakrzhevsky M. Complete Bifurcation Analysis of the Parametrically Excited Pendulum. Starptautiskā studentu zinātniski praktiskā konference „Jaunatnes loma un iespējas inženierzinātnes attīstībā”. Latvija, Daugavpils, 2011. gadā 28. aprīlī.
17. Klokov A., Pikulin D., Zakrzhevsky M. Rare Attractors in Discrete Nonlinear Dynamical Systems. Starptautiskā studentu zinātniski praktiskā konference „Jaunatnes loma un iespējas inženierzinātnes attīstībā”. Latvija, Daugavpils, 2011. gadā 28. aprīlī.
18. Beresnevich V., Klokov A. Some Characteristic Features of Nonlinear Parametric Oscillations of Flexible Elements. 2nd International Symposium RA’11 on “Rare Attractors and Rare Phenomena in Nonlinear Dynamics”, May 16 - 20, 2011, Riga - Jurmala, Latvia.
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20. Zakrzhevsky M., Schukin I., Frolov V., Klokov A., Yevstignejev V., Smirnova R., Pikulin D. Rare Attractors in Discrete Nonlinear Dynamical Systems. 2nd International Symposium RA’11 on “Rare Attractors and Rare Phenomena in Nonlinear Dynamics”, May 16 - 20, 2011, Riga - Jurmala, Latvia.
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23. Klokovs A., Šilvāns E., Retie atraktori svārstu sistēmu dinamikā. // Apvienotais pasaules latviešu zinātnieku III kongress un Letonikas IV kongress, Rīga, 2011. gadā 24. – 27. oktobris.
24. Klokov A., Zakrzhevsky M., Schukin I. Shilvan E. Rare and chaotic oscillating and rotating attractors in pendulum systems. // Proceedings of the 1st International school for young scientists "Nonlinear dynamics of machines" (School-NDM) and of the XVII Symposium "Dynamics of Vibroimpact Systems" ("DYVIS-2012"), Moscow-Klin, 20-26 May,2012.
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27. *Seminars: "RTU Mehānikas institūta un LNMK apvienotie semināri" (Rīga, Latvija, 10.11.2009, 30.03.2010, 21.09.2010, 22.02.2011, 19.02.2013, 04.06.2013).*

Main results of the thesis are presented in 31 scientific works.

## THE THESIS STRUCTURE

The doctoral thesis has been written in English. It contains the introduction, 6 chapters, conclusions, reference list, 3 appendices, 113 figures and 8 tables; the total number of pages: 191. The reference list contains 152 items.

**The first chapter** "Task description of the global bifurcation analysis of the pendulum systems" covers the problems of global bifurcation analysis of the typical pendulum driven systems and characterizes models used in the research. The notions of the coefficient of nonlinear elastic characteristic, awareness of which enables the readers to foresee chaotic oscillations in the system, are introduced. Moreover, the chapter considers the notions of periodic skeleton and passport of periodic regime on the Poincaré plane, which assist in performing the global analysis of the system dynamics in the given point of parameter space, as well as during the global bifurcation analysis in case one or several system parameters are changed. The list of pendulum driven models, discussed in the present thesis is such:

- The parametrically excited pendulum system with an additional linear restoring moment and with the periodically vibrating point of suspension in both directions;
- The pendulum system with the periodically vibrating point of suspension in the vertical direction;
- The pendulum with the vibrating point of suspension in the horizontal direction;
- The pendulum system with harmonic oscillations of the point of suspension at a certain angle to the horizontal;
- The pendulum with external harmonic excitation;
- Rigid body pendulum with linear spring and several equilibrium positions;
- Centrifugal pendulum vibration absorber with impact interactions;
- Driven damped pendulum model with a sliding mass and with the external periodic excited moment;
- Double pendulum with harmonic oscillations of the point of suspension in the vertical direction;
- Six body driven symmetric pendulum system with several equilibrium positions, linear dissipative forces between pendulums, additional linear elastic forces and harmonic excitation, applied to the first pendulum.

In all considered examples we have found that the novelty bifurcation theory's methods allow finding important unknown regular or chaotic attractors and new bifurcation groups with rare attractors.

**The second chapter** „Basic elements of the bifurcation theory of the pendulum systems" presents the approaches and methods that will be used for a global analysis of driven damped pendulum systems in this doctoral thesis. These approaches include the algorithms for construction of periodic skeleton (periodic passport), the method of searching the fixed points with and without taking into account the cyclic coordinates, method of complete bifurcation groups, Poincaré mappings from a line and from a contour, Cell-to Cell mapping, the method of separatrices of the saddle point, the method of continuation of bifurcation boundaries for the construction of the two-parameter bifurcation diagrams.

When applying the Poincaré map (see example in Fig.1), the found regime can be set in the form of data on the solution (orbit) order, number of loops on the projection of phase trajectory, coordinates of a fixed point and stability characteristics. For example, the line below –

P5 (3/5), FP (2.815634, -0.293778), Multipliers $\rho_1 = -0.467$, $\rho_2 = -1.813$ (1)

a “basic passport” of periodic regime (in the given point of parameter space of the system under consideration). The subharmonic regime with the period T = 5Tω has a projection of phase periodic trajectory with three loops; the coordinates of a fixed point on the Poincaré plane are presented in brackets. The unstable regime – inverted saddle – since $\rho_2$ = -1.813. It is worth pointing out that a basic passport of periodic regime was applied by T. Hayashi, J. Ueda and others in well-known works of Japanese scientists.

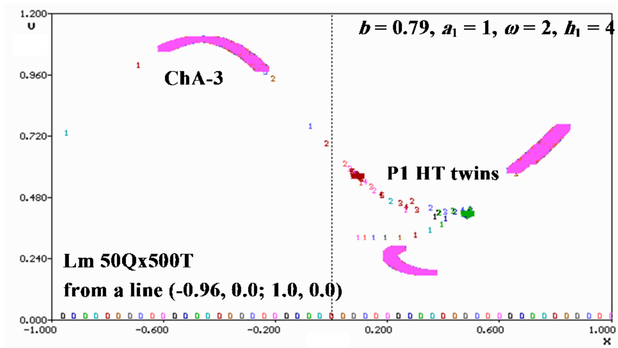
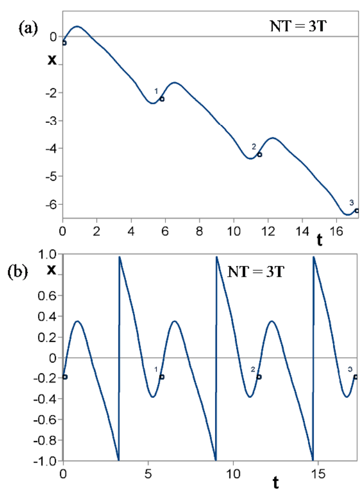



Fig. 1. Co-existence of P1 hilltop (HT) twin and chaotic ChA-3 attractors has been found in the harmonically driven damped pendulum system (1). Mapping Lm 50Qx500T from a line (-0.96, 0; 1, 0) on the Poincaré plane is shown. System parameters: $a_1 = 1$, $h_1 = 4$, $\omega = 2$, $b = 0.79$, $k = 7$.




| N | t | x | v |
|---|---|---|---|
| 0 | 0.0 | -0.177442 | 0.850614 |
| 1 | 5.711987 | -2.177443 | 0.850615 |
| 2 | 11.423973 | -4.177442 | 0.850614 |
| 3 | 17.13596 | -6.177442 | 0.850614 |
| | ... | | |
| 10 | 57.119865 | -20.177442 | 0.850614 |

| N | t | x | v |
|---|---|---|---|
| 0 | 0.0 | -0.177442 | 0.850614 |
| 1 | 5.711987 | -0.177443 | 0.850614 |
| 2 | 11.423973 | -0.177442 | 0.850615 |
| 3 | 17.13596 | -0.177442 | 0.850614 |

Fig. 2. The example of calculation of periodic rotating orbit of period-1 without (a) and with (b) taking into account the cylindrical phase space with period $L_x = 2$.

In the driven pendulum systems searching the fixed points or the rotational orbits the cylindrical phase space is taken into account with period $L_x = 2$ (Fig.2). In case of recording of cyclic coordinates the sampling period matches with the period of the external excitation force ($T = 2\pi$), and then reduced phase portrait is the phase portrait of the rotating orbit R1 with period-1

The above mentioned approaches and methods are realized in three different software NLO, ABC NDC and SPRING. All the results of numerical simulations for the pendulum systems with one and several degrees of freedom in this doctoral thesis were obtained using these software.

**In the third chapter** "Forced and parametrical oscillations of the pendulum systems with one degree of freedom" the bifurcation analysis of harmonically driven damped pendulum systems with one degree of freedom (with dimension D = 3) was performed. The effectiveness of the method of complete bifurcation groups were discussed for following pendulum driven systems: pendulum with additional linear elastic spring and vibrating point of suspension in both directions, pendulum with harmonic oscillations of the point of suspension in the vertical direction, pendulum with harmonic oscillations of the point of suspension in the horizontal direction, pendulum with harmonic oscillations of the point of suspension at a certain angle to the horizontal, pendulum with external harmonic excitation, rigid body pendulum with linear spring and several equilibrium positions, centrifugal pendulum vibration absorber with impact interactions.

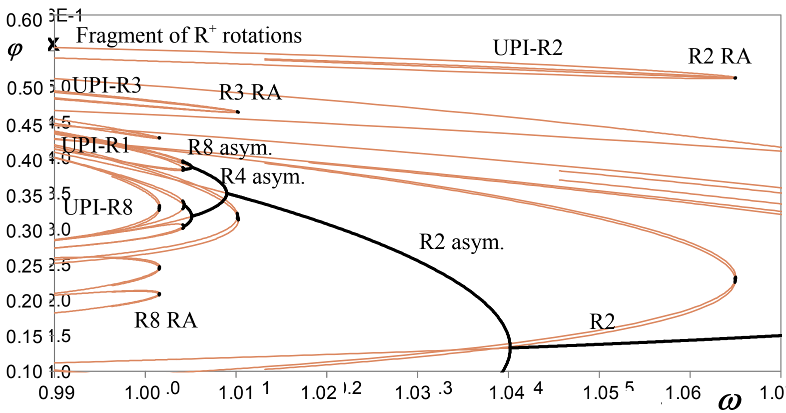



Fig. 3. Fragment of the bifurcation diagram for bifurcation groups 1T, 2T, 3T and 8T, formed by periodic rotations R1, R2, R3 and R8 in the pendulum system with the periodically vibrating point of suspension in vertical direction. The regions with the different types of chaotic rotations (UPI-R1, UPI-R2, UPI-R3 and UPI-R8) are born as a result of period-doubling bifurcations, illustrated in the figure.

The fragment of bifurcation group 1T of regime P1 zero with the period-doubling cascade, leading to the formation of a subgroup with an infinite number of unstable periodic solutions (UPI), is shown in Fig. 3. The presence of a subgroup with UPI indicates the existence of chaotic attractor or chaotic transient in this bifurcation group.

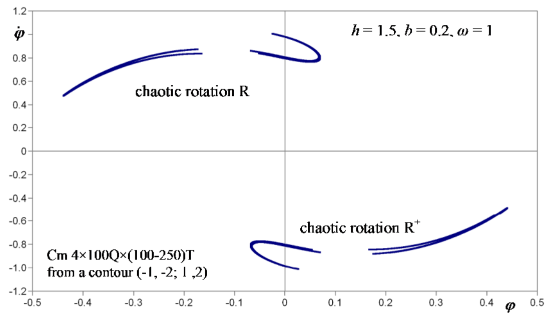



Fig. 4. Co-existence of chaotic attractors. A fragment of contour mapping Cm 4×100Q×(100-250)T on the Poincaré plane taking into account cyclicality.

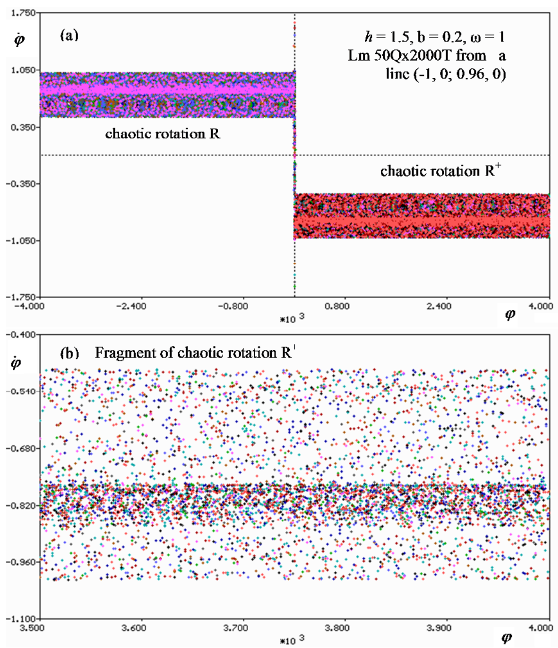



Fig. 5. Co-existence of chaotic attractors. Mapping from line Lm 50Q×2000T on the Poincaré plane without taking into account cyclicality.

For cross-section of bifurcation diagram (Fig. 3) for $\omega = 1$ co-existence of chaotic rotations is shown in a fragment of contour mapping Cm 4×100Q×(100-250)T on the Poincaré plane (Fig.4) taking into account cyclicality for the pendulum system with the periodically vibrating point of suspension in vertical direction. The same chaotic rotational attractors are shown in Fig.5 by mapping from a line Lm 50Q×2000T on the Poincaré plane without taking into account cyclicality.

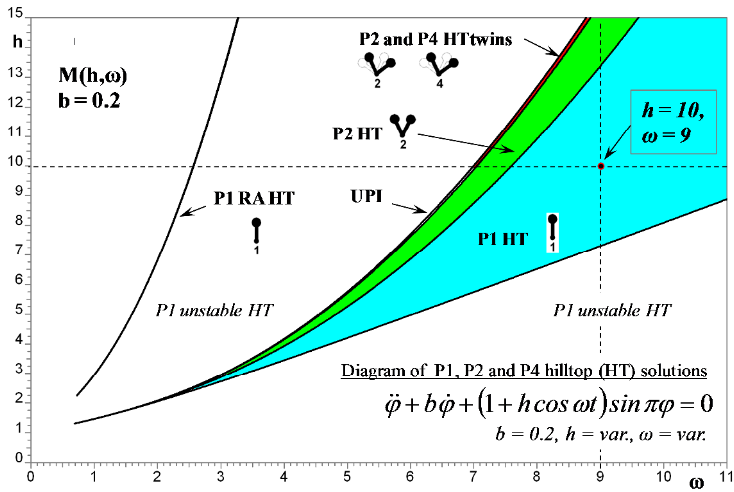



Fig. 6. Example of constructing two-parameter bifurcation diagrams in the plane ($\omega$, $h$) for the pendulum system with the vibrating point of suspension in the vertical direction. Bifurcation diagrams of regime P1 hilltop with the cascade of period-doubling bifurcations are also shown in the diagram.

The two-parameter bifurcation diagram shows the results of the research on oscillations with respect to the upper unstable equilibrium position with a period of excitation (Fig. 6). Bifurcation borders of regime P1 with the respective cascade of period-doubling bifurcations are also shown in this diagram.

The results received by the method of complete bifurcation groups in the pendulum system with the point of suspension vibrating at a certain angle $\alpha$ with respect to the horizontal show that all bifurcation groups with rare attractors are of a tip kind and each of them has not only periodic attractors, but chaotic attractors as well. Fig.7 shows, that the direction $\alpha$ of the vibrating support is sufficient for global topology of the steady-state solutions of the system. Namely some subharmonical isles with rare attractors exist only for a case, where the support is vibrating under some angle $\alpha$ from 0.044 to 0.095.

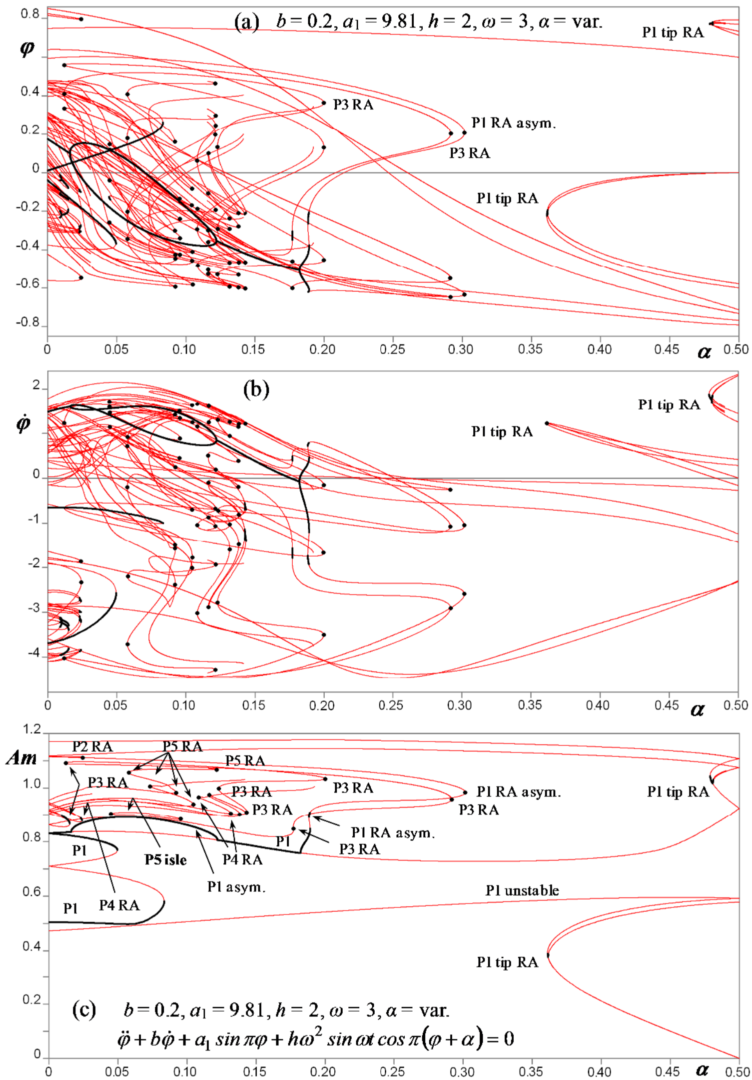



Fig. 7. Complex bifurcation diagrams for 15 different bifurcation groups for a pendulum system with the point of suspension vibrating at a certain angle α with respect to the horizontal. Many rare attractors, marked with black circles, have been found in the system. Inside each circle, there is a rare bifurcation subgroup with the period-doubling cascade and rare chaotic attractors.

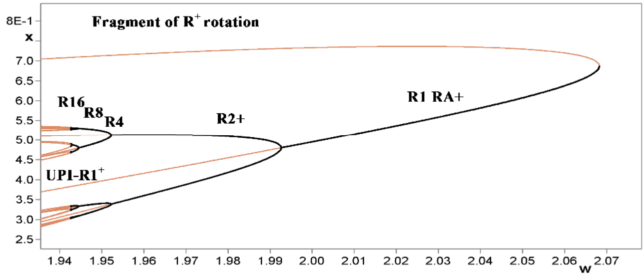



Fig. 8. Formation of chaotic rotation through the cascade of the period doubling bifurcations is shown by the rotating orbits on bifurcation diagram S1($\omega$) of the fixed periodic points of the coordinate $x$ versus frequency $\omega$ of excitation force. There is 1T bifurcation group (with R1, R2, R4, R8 and R16 rotations) of rotating regimes with complex protuberances, tip RAs and UPIs.

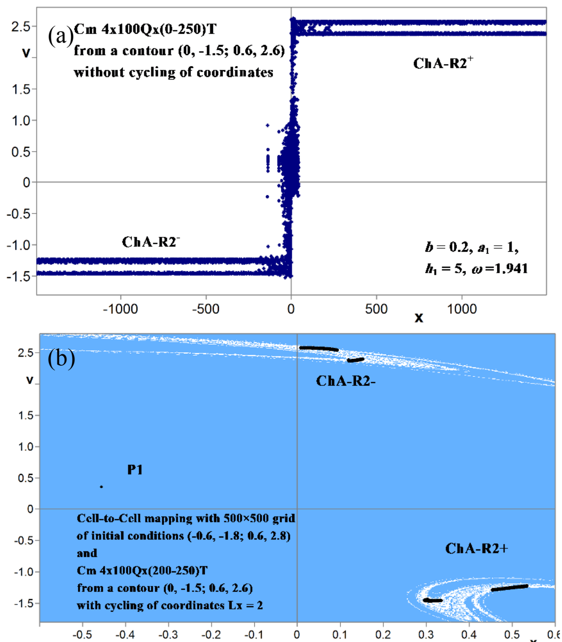



Fig. 9. Coexistence of usual P1 attractor and subharmonic rotational ChA-2 twin attractor: (a) Poincaré map Cm 4x100Qx(0-250)T from a contour (0, -1.5; 0.6, 2.6) without taking into account the cylindrical phase space with period Lx = 2; (b) basins of attraction obtained by the Cell-to-Cell mapping with 500×500 grid of initial conditions (-0.6, -1.8; 0.6, 2.8) together with Poincaré map Cm 4x100Qx(200-250)T from a contour (0, -1.5; 0.6, 2.6) taking into account the cycling of coordinate.

Fig. 9 demonstrates the coexistence of usual P1 attractor and subharmonic rotational ChA-2 twin attractor on Poincaré map and in basins of attraction.

The birth of the previously unknown rare attractors has been shown for different harmonically driven damped systems, and new bifurcation groups with complex protuberances have been obtained. The new types of interaction of different oscillating and rotating orbits have been found as well as rare and chaotic rotational regimes. Also the process of formation of chaotic rotation through the cascade of period-doubling bifurcations for different groups has been studied within the framework of the research.

**In the fourth chapter** "New bifurcation groups and rare attractors of the pendulum systems with two degrees of freedom" it is demonstrating how the method of complete bifurcation groups is applied to the global analysis of studied pendulum systems with two degrees of freedom (with dimension D = 5). Among them are pendulum system with a sliding mass and with the external periodic excited moment, and double pendulum with the periodically vibrating point of suspension in vertical direction. The main aims of the research are to investigate the qualitative behavior of the pendulum systems with 2ODF by varying the parameters of the systems and to obtain the new qualitative results of topology of bifurcation groups with rare regular and chaotic attractors.

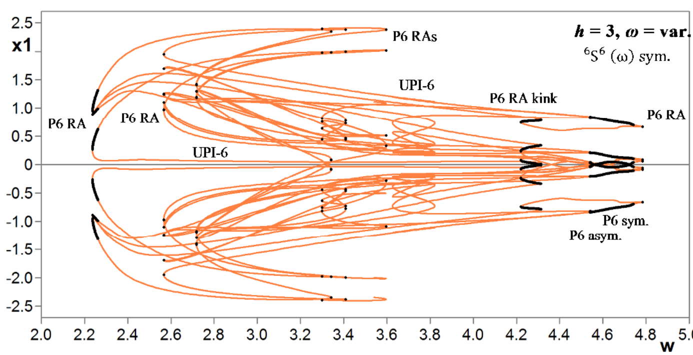



Fig. 10. Bifurcation diagrams S6($\omega$) of the first pendulum with varying frequency $\omega$ of excitation force. There is 6T symmetric isle bifurcation group with complex protuberances, with their own rare attractors of different types and UPIs.

There are complex protuberances with many rare regular attractors of different types, chaotic transients and chaotic motions. The example is shown by the 6T symmetric isle bifurcation group in the double pendulum with the periodically vibrating point of suspension in vertical direction (Fig. 10).

**In the fifth chapter** "Analysis of forced oscillations in the pendulum systems with several degrees of freedom" the most commonly studied model of forced oscillations in the driven damped pendulum systems with several degrees of freedom was investigated.

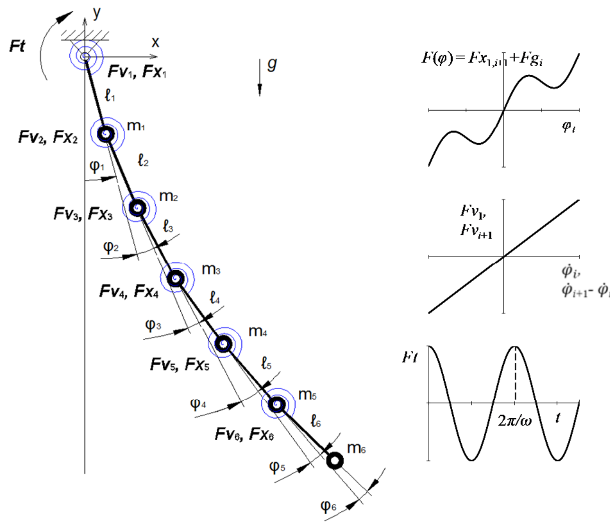



Fig. 11. Six body driven symmetric pendulum system with several equilibrium positions, linear dissipative forces $Fv_{1,i+1}$ between pendulums, pendulum's restoring forces $Fg_i$, additional linear elastic forces $Fx_{1,i+1}$ and harmonic excitation $Ft$, applied to the first pendulum.

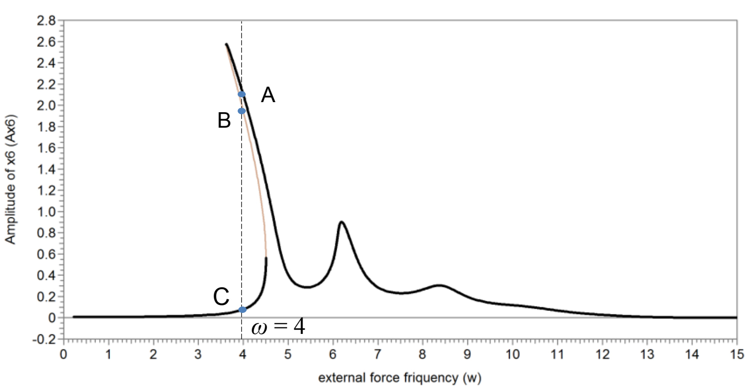



Fig. 12. Example of the 1T bifurcation group with hysteresis effect in the six body driven symmetric pendulum system.

The possibility of using the method of complete bifurcation groups for the global analysis of the six body symmetric driven pendulum systems with several equilibrium positions is demonstrated in the Fig.12. The possible applications of six body symmetric driven pendulum systems with several equilibrium positions are mixing (liquids, washing machines), robotics, communication and orientation systems of the spacecraft (antenna), etc.

**The six chapter** "Experimental investigations and animation of the pendulum systems by the method of complete bifurcation groups" shows the possibility of performance of the experimental investigations in more realistic models of pendulum systems, which were discussed in the present doctoral thesis. For these purposes, the experimental setup for natural experimental investigations in the simplest pendulum system with the periodically vibrating point of suspension in vertical direction was developed and produced (Fig.13).

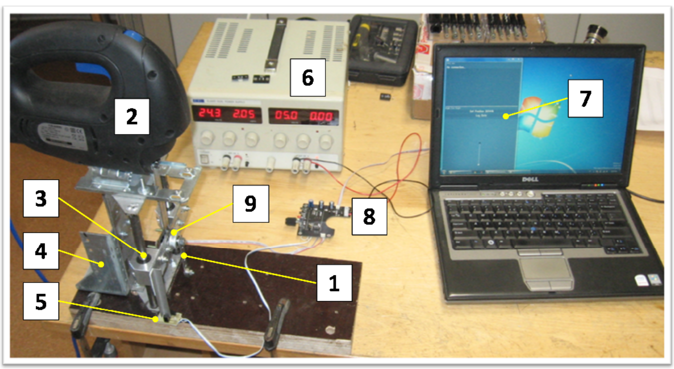



Fig. 13. General view of experimental setup of the pendulum with vertical vibrations of the support consisted of 1-pendulum; 2-exciter; 3-slides with special linear bearings; 4-frame; 5-optocoupler for measuring the frequency of excitation; 6-power supply, 5V; 7-software for experimental data collection and animation; 8-AVR microcontroller; 9-pendulum's support by ball bearing and Hall-effect 360° angle position encoder with a 14-bit high resolution output and measuring accuracy 0.05.

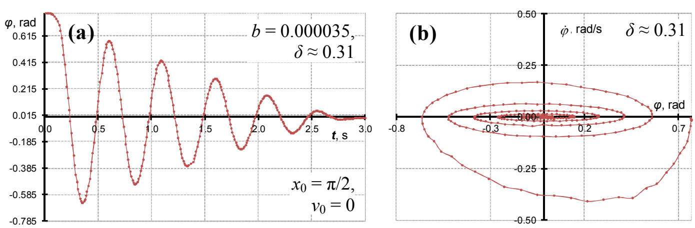



Fig. 14. Free damped oscillations of the experimental setup (Fig. 13) of the pendulum model with the periodically vibrating point of suspension in vertical direction.

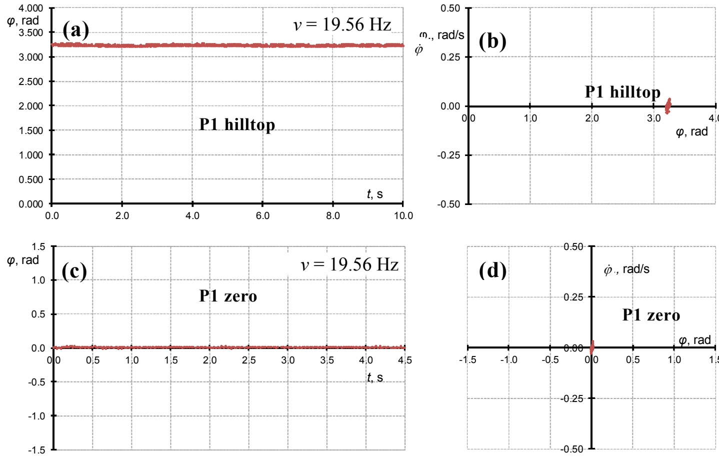



Fig. 15. Examples of periodic P1 hilltop and P1 zero regimes obtained by the experimental investigations for frequency $\nu$ = 19.56 Hz of excitation. Time history and phase portrait for stable P1 hilltop regime (a)-(b) and for stable P1 zero regime (c)-(d).

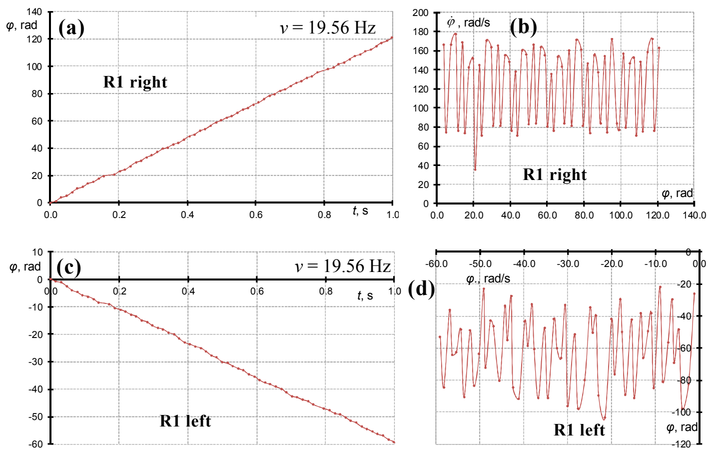



Fig. 16. Examples of periodic R1 right and R1 left rotations obtained by the experimental investigations for frequency $\nu$ = 19.56 Hz ($\omega$ = 122.9 rad/s) of excitation. Time history and phase portrait for R1 right rotation (a)-(b) and for R1 left rotation (c)-(d).

The experimental and theoretical investigations were performed for the simple pendulum driven system, in which founded regimes have qualitative agreement with the theoretical investigations. The four different oscillating and rotating regimes were found in both cases. Bifurcation diagrams also correspond to the results of numerical simulations.

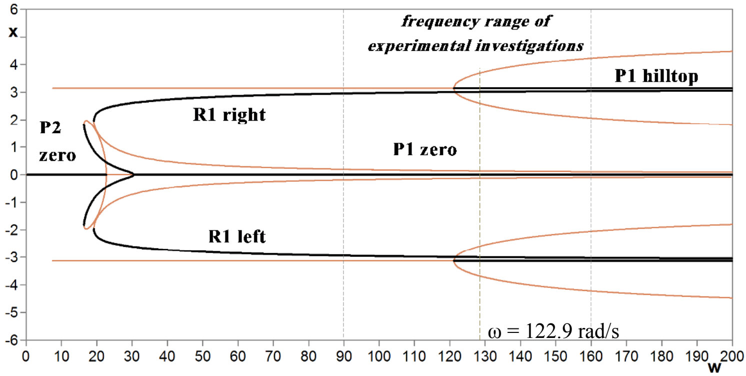



Fig. 17. Bifurcation diagram S1($\omega$) (b) shows the interaction of oscillating and rotating bifurcation groups. There is the coexistence of four different bifurcation groups with regimes P1 zero, P1 hilltop, R1 left and R1 right (see Fig. 15-16).

The visualization possibility of the founded regimes by the method of complete bifurcation groups in the pendulum systems is shown as well. It is implemented by the animation software "Parametrically Excited Pendulum" created by the author using Pascal programming language.

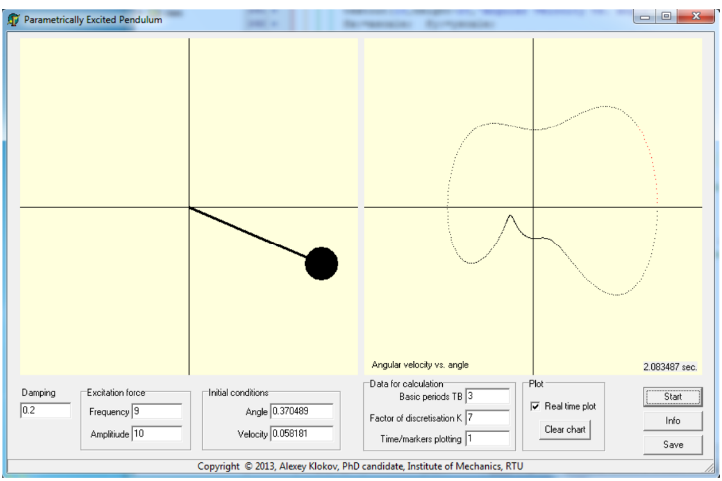



Fig. 18. The animation software "Parametrically Excited Pendulum" for the pendulum system with the periodically vibrating point of suspension in vertical direction.

The animation of nonlinear phenomena of pendulum systems can be useful not only for student as methodological material, but also for engineers who are working with real pendulum-like systems.

**In the appendices** the list of the open problems, which have not been completely solved in the present doctoral thesis, is presented. But these questions, of course, should be solved in future investigations.

## CONCLUSIONS

In the present doctoral thesis such main results were obtained:

1. Apparently, rare regular and chaotic rotational regimes have been found in the pendulum systems with or several degrees of freedom for the first time within the framework of this research. These nonlinear phenomena define a great theoretical and practical interest of studies of nonlinear dynamics in the pendulum systems.
2. It is shown that the use of the method of complete bifurcation groups allows conducting the qualitative global bifurcation analysis of the pendulum systems and finding the new bifurcation groups and previously unknown regular and chaotic oscillating, oscillating-rotating and rotating orbits (regimes, attractors) by the example of the simplest systems with one or more degrees of freedom. The qualitative topology of different bifurcation groups is investigated and new periodic and chaotic orbits are found.
3. The birth of the previously unknown rare attractors has been shown for different harmonically driven damped systems using the method of complete bifurcation groups, and new bifurcation groups with complex protuberances have been obtained. The new types of interaction of different oscillating and rotating orbits have been found as well as rare and chaotic rotational regimes.
4. Apparently, the process of formation of chaotic rotation through the cascade of period-doubling bifurcations for different groups has been found for the first time and studied within the framework of the research.
5. The interaction of different bifurcation groups in the pendulum systems with one or several degrees of freedom is investigated. Two-parameter bifurcation diagrams were constructed. It is shown, that the existence of bifurcation subgroup with unstable periodic infinitium (UPI) always leads to chaotic behavior of the system: chaotic attractor or transient chaos
6. The main qualitative results of topology of bifurcation groups with rare regular and chaotic attractors for the pendulum systems with two degrees of freedom were investigated. The subharmonic, chaotic and rare periodic behaviour were investigated. There are complex protuberances with many

rare regular attractors of different types, chaotic transients and chaotic motions.

7. The possibility of using the method of complete bifurcation groups for the global analysis of the six body symmetric driven pendulum systems with several equilibrium positions is demonstrated.
8. The possibility of performance of the experimental investigations in more realistic models of pendulum systems, which were discussed in the present doctoral thesis, is shown. The visualization possibility of the founded regimes by the method of complete bifurcation groups in the pendulum systems is investigated as well.

Thus, in the doctoral thesis it is was shown, that the method of complete bifurcation groups allows conducting the global bifurcation analysis of the pendulum systems with one or several degrees of freedom and finding new nonlinear phenomena, new bifurcation groups, unknown before rare periodic and chaotic attractors both oscillating and rotating. The possibility of usage of new obtained results in real objects is discussed as well.

The author is of the opinion that the method of complete bifurcation groups will be useful for global bifurcation analysis and for search of rare attractors as well as for other pendulum systems.

## LIST OF PUBLICATIONS AND PATENT

Results of the thesis are published in 31 scientific works: 4 scientific monographs, 6 papers in international journals, 20 research papers in the proceedings of the international scientific conferences and 1 patent:

1. Klokov A., Viba J. Modeling of mixed excitation system // Scientific Journal of RTU. 6. series., Transport and Engineering. - 28. vol. (2008), pp. 55-62. [in Latvian].
2. Klokov A., Zakrzhevsky M. Rare attractors in the spacecraft dynamics. // XVI Symposium “Dynamics of Vibroimpact Systems” (“DYVIS-2009”), Moscow, Zvenigorod, 2009, pp. 183-192. [in Russian].
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18. Zakrzhevsky M., Schukin I., Frolov V., Klokov A., Yevstignejev V., Smirnova R., Pikulin D. Rare Attractors in Discrete Nonlinear Dynamical Systems. // Proceedings of the 2nd International Symposium RA'11 on "Rare Attractors and Rare Phenomena in Nonlinear Dynamics", May 16 - 20, 2011, Riga - Jurmala, Latvia, pp. 21-25.
19. Beresnevich V., Klokov A. Some Characteristic Features of Nonlinear Parametric Oscillations of Flexible Elements. // Proceedings of the 2nd Intern. Symposium RA'11 on "Rare Attractors and Rare Phenomena in Nonlinear Dynamics", May 16-20, 2011, Riga - Jurmala, Latvia, pp. 57-64.
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21. Beresnevich V., Tsyfansky S., Klokov A. Nonlinear lateral oscillations of flexible elements in machines: characteristic features and practical applications. // Springer Proceedings in Physics with title "Vibration Problems ICOVP 2011: The 10th International Conference on Vibration Problems", Vol. 139 (2011), pp. 75-80.

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23. Kremer E., Zakrzhevsky M., Klokov A. Complete Bifurcation Analysis with Rare Attractors of a Pendulum Vibration Absorber // Proceedings of the 4th IEEE International Conference on Nonlinear Science and Complexity “NSC 2012”, August 6-11, 2012 Budapest, Hungary, pp. 205-210.
24. Yevstignejev V., Klokov A., Smirnova R., Schukin I. Rare Attractors in Typical Nonlinear Discrete Dynamical Models // Proceedings of the 4th IEEE International Conference on Nonlinear Science and Complexity “NSC 2012”, August 6-11, 2012, Budapest, Hungary, pp. 229-234.
25. Zakrzhevsky M., Smirnova R., Schukin I., Yevstignejev V., Frolov V., Klokov A., Shilvan E. *Nonlinear Dynamics and Chaos. Bifurcation Groups and Rare Attractors*. – Riga: RTU Publishing House, 2012 – 181 p. [in Russian].
26. Zakrzhevsky M.V., Pikulin D.A., Klokov A.V., Smirnova R.S., Yevstignejev V.Yu., Schukin I.T.. *Rare attractors in discrete nonlinear dynamical systems*. – Riga: RTU, 2013. – 105 lpp.
27. Zakrzhevsky M. and Klokov A. *How to find rare attractors and chaos in nonlinear dynamical systems by the method of complete bifurcation groups. The collection of bifurcation diagrams in the pendulum driven systems*. – Riga, 2013 – 310 p.
28. Zakrzhevsky M., Schukin I., Yevstignejev V., Frolov V., Smirnova R., Klokov A., Shilvan E. *Nonlinear Dynamics and Chaos. Complete Bifurcation Analysis and Rare Attractors*. – Riga: RTU Publishing House, 2013. – 210 p. (in print).
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Submitted LR patent:

31. Patent Nr. P-13-90, Latvian Republic. „Testing vibration stand with complex stationary and chaotic oscillations” / Klokov A., Zakrzhevsky M., Yevstignejev V., Schukin I., Shilvan E., 2013.

# LIST OF PROJECTS

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2. LZP grants 2010. g. - 2012. g. Nr. 09.1587, „Retie atraktori un reto dinamisko parādību fundamentāla teorija un ultraskaņa viļņu pielietošana”.

3. International project with German company “LuK GmbH & Co. OHG”. 2012 - 2013. “The development, adaptation, providing and support of universal software Spring and ABC NDC for global analysis of strongly nonlinear dynamical systems”, Nr.11/MI–2012.

# GLOSSARY

This glossary contains a list of some basic definitions and terms of Bifurcation Theory. Terms, which are proposed by the research group “Nonlinear dynamics, chaos, catastrophes and control” of the Institute of Mechanics of RTU under the guidance of Prof. M. Zakrzhevsky, are marked with a sign (*).

**Andronov-Hopf bifurcation** – a birth of quasi-periodic oscillations from the periodic one by varying parameter of dynamical system.

**Attractor** – a point or collection of points in the phase space, where all the initial states tend to approach the steady-state. There are several types of attractors, such as a point attractor, a periodic or quasi-periodic attractor, chaotic attractor or rare attractor.

**Basin of attraction** – a collection of points in the phase space (initial conditions), from which a concrete regime (attractor) is realized after a transient process.

**Bifurcation** – a qualitative change in the topology of the phase space by varying a bifurcation parameter of dynamical system.

**Bifurcation analysis** – an analysis of behaviour of a dynamical system by varying a bifurcations parameter.

**Bifurcation analysis (complete)*** – complete analysis of behaviour of a dynamical system by varying a bifurcations parameter, whereby complete bifurcation groups are constructing.

**Bifurcation group (complete)*** – all stable and unstable branches of periodic orbits, the branches of which are connected to each other at a bifurcation points.

**Bifurcation parameter** – a parameter of a dynamical system, by varying of which a bifurcation occurs.

**Chaos** – irregular oscillations of a deterministic system, which have a high sensitivity to initial conditions.

**Chaotic attractor** – the same as *chaos.*

**Dynamical system (DS)** – a dynamic continuous or discrete model, behavior of which unambiguously and deterministically depends on it structure (S), parameters (P) and states (initial conditions $Q_0$).

**Equilibrium state** – see *fixed point.*

**Fixed point (FP)** – a singular point on Poincaré plane, which corresponds to stable or unstable periodic solutions, i.e. to closed phase trajectories. There are four types of fixed points: node, focus, saddle and center.

**Fold bifurcation** – a bifurcation, where stable and unstable branches of a same regime mutually eliminates.

**Initial conditions** – a vector of generalized coordinates (displacement, velocity, initial phase of external influence), which is used to describe a mathematical model of a dynamical system at the initial time $t_0$.

**Island*** – a closed branch of periodic regime in a limited region of varied parameter in the bifurcation diagram.

**Linear dissipation** – a friction force is proportional to the velocity.

**Method of Complete Bifurcation Group (MCBG)*** – a complex of approaches used for the global bifurcation analysis of dynamical systems for variable parameter values. MCBG is applied to find all stable and unstable periodic solutions for given parameter and phase space region.

**Multiplicity** – a nonlinear phenomenon of coexistence of different stationary regimes (attractors) in phase space with the same parameters of the system and the invariance of its structure. An implementation of concrete regime depends on initial conditions.

**Node** – an equilibrium state, which corresponds to the stability or instability of particular fixed point of the phase plane. Roots of the characteristic equation (multipliers) – real negative (positive) numbers.

**Passport of a periodic orbit*** – includes the information about the order of regime, initial conditions or coordinates of its fixed points, multipliers characterizing the stability of found regimes, number of loops in phase projection closed curve and some additional information, if necessary. Data included in periodic skeleton allow complete reconstruction of the regime, preserving all its characteristics.

**Period doubling bifurcation** – a local bifurcation in which a limit cycle of the system changes into a cycle of twice the period as a bifurcation parameter is varied.

**Periodic regime** – a stable or unstable periodic solution of differential equation or closed phase trajectory, which are assigned by fixed points of the corresponding T- mapping.

**Periodic skeleton*** – all stable and unstable periodic regimes with their passport for all bifurcation groups and UPI subgroups found for the fixed parameter value.

**Phase portrait** – a portrait of regime, structure of partitioning of the phase space on trajectories for regime in studied dynamical system.

**Poincaré mapping** – see *T-mapping.*

**Poincaré section** – section, obtained by a strobe of dynamical variables (displacement $x_p$ and velocity $\dot{x}_p = v_p$) after a period of excitation force $T_\omega$.

**Protuberance*** – on the bifurcation diagram- the set of stable and unstable branches of periodic regimes, corresponding to one bifurcation group, restricted by period doubling or some other bifurcations of the same order.

**Quasi-periodic oscillations** – oscillations with two or more incommensurable frequencies.

**Regular oscillations** – regular attractors (except chaotic one): point, periodic, quasi-periodic.

**Rare attractors*** – periodic regime, which exist, as a rule, in quite narrow ranges of system varied parameters. The types of rare attractors can be such: tip, egg-like or dumbbell, kink or hysteresis, small isolated island (isola isle).

**Regime** – characteristic of dynamical process. There are stationary and transient regimes.

**Saddle** – an equilibrium state, which corresponds to an unstable fixed point of particular phase plane. Roots of the characteristic equation (multipliers) – real numbers, one of which is in absolute value greater than one.

**Separatrix** – stable and unstable manifolds (invariant curves) passing through a singular point of the saddle.

**Simple island*** – island, which has one stable and one unstable branch.

**Stationary chaos** – chaotic behaviour of the system that was established after the decay of transients in a dissipative dynamical system, the same as chaotic attractor.

**Symmetry breaking bifurcation** – a global bifurcation in which a symmetrical attractor is replaced by two coexisting asymmetrical attractors (twins).

**T-mapping** – mapping, obtained on the Poincaré section by a strobe of dynamical variables (displacement $x_p$ and velocity $\dot{x}_p = v_p$) after a period of excitation force $T_\omega$.

**Transient chaos** – chaotic behavior that was established in the system during the transition to some regular stationary regime.

**Transient process** – nonstationary process that is describing the motion of the system from an initial state at time $t_0$ during time $t$ of the transition to some steady state.

**Twins*** – periodic regimes, which appear in symmetrical systems as a result of symmetry breaking bifurcation.

**Unstable periodic infinitium (UPI)*** – bifurcation subgroup with the infinite number of unstable periodic regimes corresponding to $nT$ bifurcation group.