# Determination of the Optimal Value of the Radius of a Circular Cylindrical Post in a Rectangular Waveguide for Measurement of the Dielectric Permittivity

**Roman Kushnin and Janis Semenjako**
Riga Technical University, Riga, Latvia

**Abstract**— The goal of this paper is to determine the optimal value of the radius of a circular cylindrical post in a rectangular waveguide, that is, such a value of the radius of the post at which measurement uncertainty takes minimum value. To that end, we employ the well-known Monte Carlo method. Probability distributions for input quantities of the Monte Carlo model are assigned according to the principle of maximum entropy.

## 1. INTRODUCTION

This paper is inspired by a recently published paper [1], in which a method for determination of the optimal radius of a dielectric post in a cylindrical metallic resonator for measurement of the dielectric permittivity has been proposed. This task requires a lot of computation time, since its based upon the Monte Carlo method that, in turn, requires a very large number of trials in order for estimation of the measurement uncertainty to be reliable and each of these trials requires solution of the inverse scattering problem, that in our case is being solved by iteratively minimizing the distance between simulated and measurement data. In order to accomplish this task in a reasonable amount of time, we need a very fast approach for solving the direct scattering problem, since the computation time required to solve the inverse scattering problem is proportional to that required by an algorithm for solving the corresponding direct scattering problem. A simpler way to evaluate measurement uncertainty is to follow GUM framework. Unfortunately, validity of the GUM framework is restricted only to models that lend themselves to an adequate linear approximation.

The problem of scattering of the dominant mode in a rectangular waveguide by a cylindrical post was first treated in Notes on lectures by Julian Schwinger [2] by using the variational method. Numerical results obtained by application of this method also may be found in [3]. Unfortunately, it has been found that results obtained by the variational method are highly inaccurate for posts of large electric radius, particularly near resonance frequencies. An attempt to improve the accuracy of the variational method was made in [4] by using the so-called second order approximation, that is, by using one more term in both series expansions instead of using just one term. Also many other approaches have been proposed by various authors over the last several decades [5–9].

A first approach based on division of geometry of a problem into subregions of simple regular geometry has been proposed by Nielsen [10]. Since the fields in regular regions are expressed in terms of a series of solutions of the homogenous Helmholtz equation, it allows one to solve the boundary problem on the interfaces between layers of the post analytically that, in turn, considerably reduce computational efforts as well as provides sufficiently high resolution of the field distribution. In order to find unknown expansion coefficients, infinite series are truncated and then by applying the point matching procedure the system of linear equations is set up and solved for unknown expansion coefficients. Unfortunately, this approach converges to the correct result only for post of small electric radius (the product of the radius of the post and wavenumber). It was found that this limitation can be overcome by replacing the rectangular interaction region with the circular one, where the center of the circular interaction region (the region where point-matching procedure applies) coincides with the axis of the post and its radius is equal to one-half the width of the broader wall of the waveguide [11]. It was also found that applying numerical integration on the surface of the interaction region instead of the point matching procedure yields faster convergence [12].

## 2. DISCUSSION

Analytical solution of the inverse scattering problems is possible only for very simple problem geometries. Even when the direct scattering problem may be solved analytically, it may not be possible to solve the corresponding inverse problem analytically without any approximations. Due to this fact, some numerical procedure has to be used. One of the most common and general ways to solve this problem numerically is to convert the inverse scattering problem into an equivalent numerical optimization problem, where an objective function is typically chosen as the distance between calculated and measured values of $S$ parameters.

$$Q(f) = \sqrt{\sum_{m=1}^{2}\sum_{n=1}^{2}(\log S_{nm}^{e} - \log S_{nm}^{s})^2} \tag{1}$$

where $Q(f)$ — is the objective function; $S_{nm}^{e}$ — measured values of scattering matrix entries; $S_{nm}^{s}$ — values of the scattering parameters obtained by solving the corresponding direct scattering problem. It is obvious that the objective function takes minimum value when values of the coordinates correspond to the solution of the inverse scattering problem. There are many algorithms for minimization of objective functions, but in this study we use a simple pattern search method [13].

In case when measurements are made only at one particular frequency the solution of the problem is not unique, since the coefficients may take the same value at different values of the complex permittivity. This issue may be overcome by making measurements at two different frequencies. Nevertheless, this multi-frequency method cannot be applied in a case of highly dispersive materials, where constitutive parameters changes very rapidly with frequency. In other words, without at least approximate knowledge of this dependence, it is most likely that error in the results will be large. In such cases, we have to use other measurement methods that not only ensures uniqueness of the solution, but also allows one to make all measurements at some fixed frequency. One such method is to make measurements for two samples with different values of some dimension at a fixed frequency value. Another commonly used method is to use a movable short circuit at one of ports of the measurement cell. Unfortunately, in this case it is possible to measure only the reflection coefficient, the absolute value of which in a case of lossless sample is always equal to unity that means that in a lossless case the only quantity we can measure is the phase of the reflection coefficient.

## 3. THE DIRECT SCATTERING PROBLEM

Throughout this study it is assumed that the walls of the rectangular waveguide are perfectly conducting. Also we assume that only dominant mode $TE_{10}$ may propagate in the waveguide and all other modes are evanescent and exhibit very rapid exponential decay with distance from the sample. Both these assumptions lead to simplification of the numerical analysis. In order to solve the problem we divide the waveguide into three separate regions as depicted in Figure 1. In region I and III scattered fields are represented in terms of waveguide modes.

$$E_y^I = \sum_{m=1}^{\infty} A_m \cos\frac{m\pi x}{a} e^{-jk_m z} \tag{2}$$

$$E_y^{III} = \sum_{m=1}^{\infty} B_m \cos\frac{m\pi x}{a} e^{jk_m z} \tag{3}$$

In region II fields and inside the cylindrical sample fields are represented in terms of cylindrical waves.

$$E_y^{II} = \sum_{n=0}^{\infty}(C_n J_n(k_o r) + D_n Y_n(k_o r))\cos n\Theta, \tag{4}$$

$$E_y^{p} = \sum_{n=0}^{\infty} E_n J_n(k_p r)\cos n\Theta, \tag{5}$$

where $k_m$ — is the waveguide wavenumber, $k_o$ — wavenumber in the air, $k_p$ — wavenumber in the material of the sample. Expressions for corresponding magnetic fields in these regions may be obtained by using the second Maxwell's equation.

Enforcing boundary conditions on the surface of the post as well as taking the advantage of the mutual orthogonality of cylindrical waves with respect to azymuthal coordinate $\Theta$ and eliminating the unknown constant $E_n$, the expression relating unknown constants $D_n$ and $C_n$ may be obtained as follows.

$$D_n = \alpha_n \cdot C_n \tag{6}$$

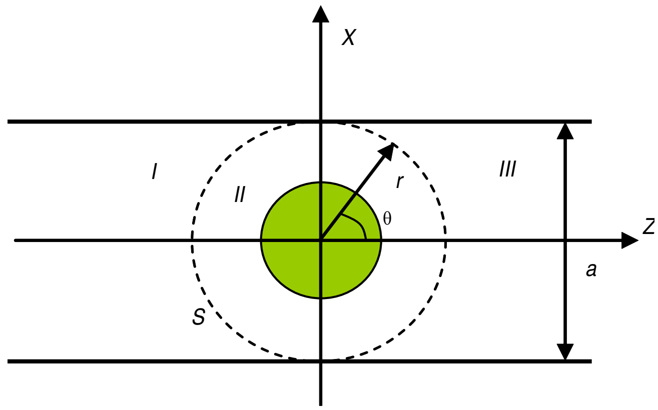


Figure 1: The rectangular waveguide containing the cylindrical dielectric sample.

where

$$\alpha_n = \frac{k_p J_n'(k_p r_p) J_n(k_o r_o) - k_o J_n(k_p r_p) J_n'(k_o r_p)}{k_o J_n(k_p r_p) Y_n'(k_o r_p) - k_p J_n'(k_p r_p) Y_n(k_o r_p)}$$

By enforcing boundary conditions on the surface of the interaction region $S$ we obtain the set of four equations. As it can be seen, this set contains infinite series of cylindrical waves and waveguide modes. However, it is impossible to solve this system by any analytical method, since cylindrical waves and waveguide modes are not mutually orthogonal. It means that we are forced to resort to the truncation of infinite series of waveguide modes and cylindrical terns, thereby converting it into the system of linear algebraic equations. Application of the truncation procedure yields finite series of waveguide modes and cylindrical waves with only $M$ and $N$ first terms retained, respectively. Projecting the difference between field representations on both sides of the interaction region $S$ upon the set of test functions $\cos(p \cdot \Theta)$ (where $p = 1, 2, \ldots, M$) and carrying out a number of simple algebraic manipulations the final system of linear algebraic equations for two sets of unknown coefficients $A_m$ and $B_m$ is obtained.

## 4. NUMERICAL RESULTS

Among the factors influencing the measurement accuracy are the limited resolution, residual systematic error, connection mismatch, and geometrical imperfections of the sample, such as a small shift in the position of the sample and the accuracy of the measurement of the radius of the cylindrical sample. Since the influence of different sources of uncertainty on the uncertainty of the output quantity may be dependent on values of parameters of the model it may be possible that for some optimal combinations of values of these parameters the standard uncertainty $u(\varepsilon)$ of the output quantity will be smaller than for all other combinations. In this paper we consider only the dependence of the standard uncertainty $u(\varepsilon)$ of the output quantity upon only one of the model parameters, namely, the radius of the cylindrical sample. In order to find optimal values of the radius of the sample we estimate the standard uncertainty in measurement of the dielectric permittivity by using the Monte Carlo method. Probability distributions for input quantities of the model are assigned according to the maximum entropy principle. In our case we take into account uncertainties due to the frequency accuracy, imperfect measurement of the radius of the sample and scattering data and assume that all these quantities are distributed according to the normal distribution. Normally distributed random numbers are generated by using uniformly distributed random numbers generated by the pseudo-random number generator and applying Box-Muller transform. Values of the dielectric constant can be extracted from scattering data by means of a pattern search optimization algorithm [11], using the objective function (1).

Application of the Monte Carlo method for each value from some discrete set of values of the radius of the post allows one to find optimal (minimal) values of the standard uncertainty of the dielectric constant. In the present case, for all numerical simulations we have used the number of trials equal to 100000 and values of the first and second measurement frequencies equal to 2.4 GHz and 2.6 GHz, respectively. The results of Monte Carlo simulations for three different types of extraction of the dielectric constant from measured values of reflection data of the field reflected by the cylindrical dielectric post with $\varepsilon = 5$ inserted in the rectangular waveguide with width of the broader wall a equal to 100 mm are depicted in Figures 2–4. The results obtained by numerical simulation of measurement of both the absolute value and phase of the reflection coefficient are depicted in Figure 2. In turn, the results obtained by numerical simulation of measurement of only the absolute value and phase of the reflection coefficient are depicted in Figure 3 and Figure 4, respectively. In Figure 5, the results obtained for the post with $\varepsilon = 10$ inserted in the waveguide with $a = 100\,\text{mm}$ by simulating measurement of both the absolute value and phase of the reflection coefficient, are depicted. As these graphical results show, there is a variation of the standard uncertainty of measurement of the dielectric constant with the value of the relative radius of the sample $r/a$. As seen in Figures 2 and 4, the value of the standard uncertainty for certain values of $r/a$ is less than for other values.

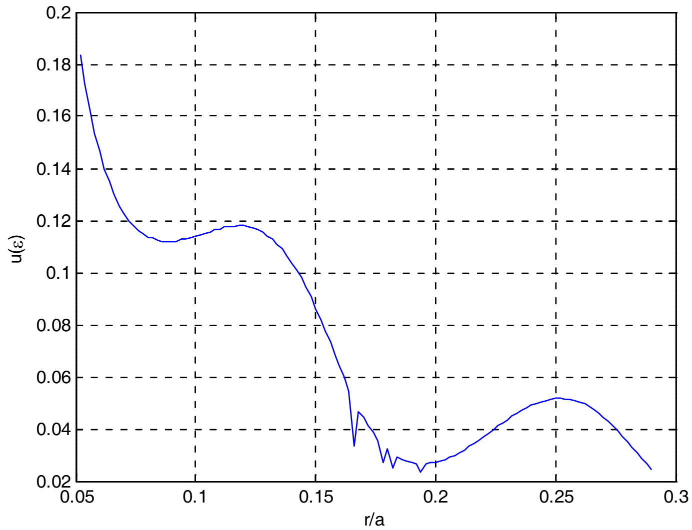


Figure 2: The standard uncertainty $u(\varepsilon)$ of the dielectric constant versus the relative radius of the post $r/a$, measuring both the absolute value and phase of the reflection coefficient with the following values of standard uncertainties of input quantities: $u(|R|) - 0.05|R|$, $u(\arg(R))$ — 1.7 degrees, $u(r)$ — 0.01 mm, $u(f)$ — 2.4 MHz.

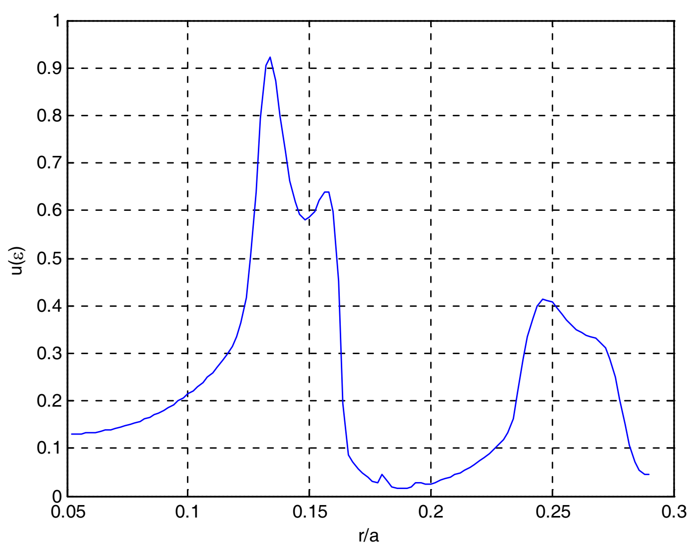


Figure 3: The standard uncertainty $u(\varepsilon)$ of the dielectric constant versus the relative radius of the post $r/a$, measuring only the absolute value $|R|$ of the reflection coefficient with the following values of standard uncertainties of input quantities: $u(|R|) - 0.05|R|$, $u(r)$ — 0.01 mm, $u(f)$ — 2.4 MHz.

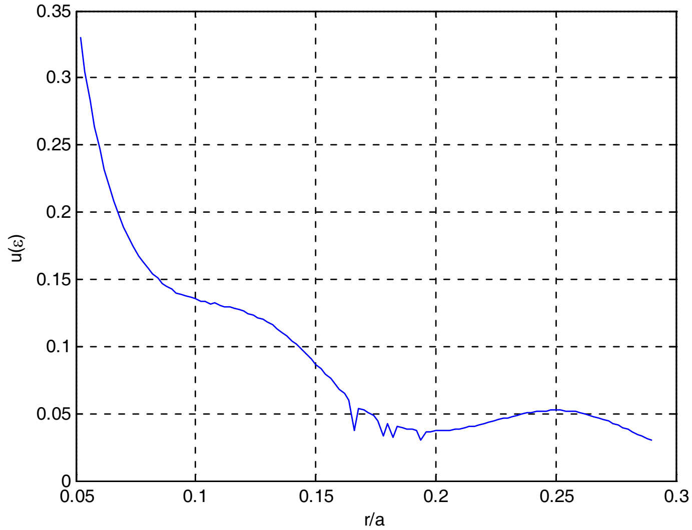


Figure 4: The standard uncertainty $u(\varepsilon)$ of the dielectric constant versus the relative radius of the post $r/a$, measuring only the phase of the reflection coefficient with the following values of standard uncertainties of input quantities: $u(\arg(R))$ — 1.7 degrees, $u(r)$ — 0.01 mm, $u(f)$ — 2.4 MHz.

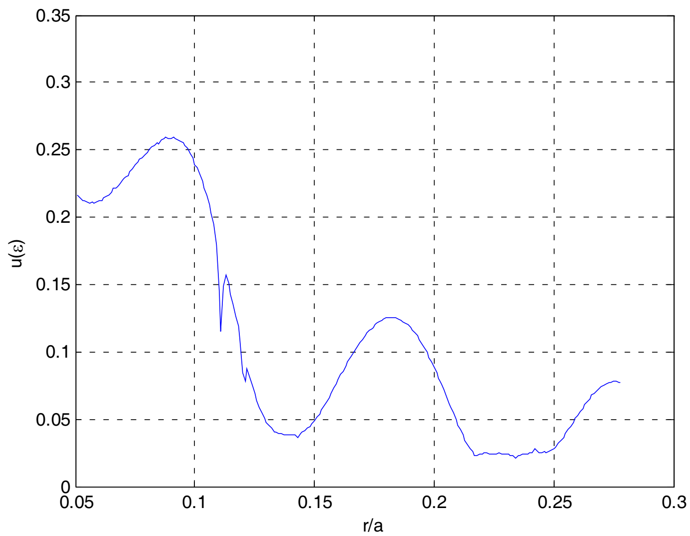


Figure 5: The standard uncertainty $u(\varepsilon)$ of the dielectric constant versus the relative radius of the post $r/a$, measuring both the absolute value and phase of the reflection coefficient with the following values of standard uncertainties of input quantities: $u(|R|) - 0.05|R|$, $u(\arg(R))$ — 1.7 degrees, $u(r)$ — 0.01 mm, $u(f)$ — 2.4 MHz.

## 5. CONCLUSIONS

We have chosen from a number of approaches that have been proposed by various authors the one that provides accurate results and at the same time provides sufficiently fast convergence. In order to find values of the radius of the cylindrical sample at which the value of measurement error has the smallest influence on the accuracy of determination of the dielectric constant, we have applied the Monte Carlo method to find values of the radius of the sample at which the standard uncertainty of the dielectric constant has its smallest value. Uncertainties due to the frequency accuracy, imperfect measurement of the radius of the sample and scattering data have been taken into account, assuming that all these quantities are distributed according to the normal distribution. Numerical results show that the standard uncertainty of measurement of the dielectric constant changes with the value of the relative radius of the sample $r/a$ and for certain values of $r/a$, the standard uncertainty is smaller than for other values.

**REFERENCES**


1. Paez, E., M. A. Azpurua, C. Tremola, and R. C. Callarotti, "Uncertainty estimation in complex permittivity measurements by shielded dielectric resonator technique using the monte carlo method," *Progress In Electromagnetics Research B*, Vol. 41, 101–119, 2012.
2. Schwinger, J., Notes on lectures, *Discontinuities in Waveguides*, Gordon and Breach, New York, 1968.
3. Marcuvitz, N., *Waveguide Handbook*, London, 1986.
4. Araneta, J. C., M. E. Brodwin, and G. A. Kriegsmann, "High temperature characterization of dielectric rods," *IEEE Trans. Microwave Theory and Techniques*, Vol. 32, 1328–1334, 1984.
5. Leviatan, Y., P. G. Li, A. T. Adams, and J. Perini, "Single-post inductive obstacle in rectangular waveguide," *IEEE Trans. Microwave Theory and Techniques*, Vol. 31, 1983.
6. Leviatan, Y. and G. F. Sheaffer, "Analysis of inductive dielectric posts in rectangular waveguide," *IEEE Trans. Microwave Theory and Techniques*, Vol. 35, 48–59, 1987.
7. Leviatan, Y. and G. F. Sheaffer, "Composite inductive posts in waveguide — A multifilament analysis," *IEEE Trans. Microwave Theory and Techniques*, Vol. 36, 779–783, 1988.
8. Ise, K. and M. Koshiba, "Dielectric post resonances in a rectangular waveguide," *IEE Proceedings, Part H — Microwaves, Antennas and Propagation*, Vol. 137, 61–66, 1990.
9. Chung, I. G. H. and A. A. Hesham, "Multiple dielectric post in a rectangular waveguide," *IEEE Trans. Microwave Theory Technuques*, Vol. 34, 883–889, 1986.
10. Nielsen, E. D., "Scattering by a cylindrical post of complex permittivity in a waveguide," *IEEE Trans. Microwave Theory Technuques*, Vol. 17, 148–153, 1969.
11. Sahalos, J. N. and E. Vafiadis, "On the narrow-band microwave filter design using dielectric rod," *IEEE Trans. Microwave Theory Technuques*, Vol. 33, 1165–117, 1985.
12. Abdulnour, J. and L. Marchildon, "Scattering by a dielectric obstacle in a rectangular waveguide," *IEEE Trans. Microwave Theory Technuques*, Vol. 41, 1988–1994, 1993.
13. Yu, W. C., "Positive basis and a class of direct search techniques," *Scientia Sinica [Zhongguo Kexue]*, 53–68, 1979.