RIGA TECHNICAL UNIVERSITY
Faculty of Computer Science and Information Technology
Institute of Applied Mathematics

IRINA EGLĪTE

SHALLOW FLOW STABILITY ANALYSIS WITH APPLICATIONS IN HYDRAULICS

DOCTORAL THESIS

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Dr. math., Prof. Andrejs Koliškins

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ABSTRACT

Linear and weakly nonlinear stability analysis of shallow mixing layers is presented in the Doctoral Thesis. The flow is assumed to be slightly curved along the longitudinal coordinate. Linear stability is analysed from temporal and spatial points of view under the rigid-lid assumption. The friction coefficient varies with respect to the transverse coordinate (the case of constant friction coefficient usually analysed in the literature is a particular case of the analysis presented in the Thesis). The corresponding linear stability problems are solved numerically using pseudo-spectral collocation method based on Chebyshev polynomials. In addition, the problem is generalized for the case of two-component shallow flows under the assumption of large Stokes numbers.

The effect of asymmetry of base flow profile on the stability characteristics is analysed. Two approaches to weakly nonlinear stability are presented as well. The first approach is based on the parallel flow assumption and can be applied for the case where the bed-friction number is slightly smaller than the critical value. Using the method of multiple scales an amplitude evolution equation is derived for the most unstable mode. It is shown that for slightly curved shallow mixing layers which contain or do not contain particles the amplitude equation is the complex Ginzburg-Landau equation. The coefficients of the equation are calculated explicitly in terms of integrals containing linear stability characteristics of the flow. Stability of plane wave solutions of the Ginzburg-Landau equation is analysed. Numerical solutions of the Ginzburg-Landau equation are presented for different initial conditions.

The second approach takes into account slow longitudinal variation of the base flow. The analysis is based on weakly non parallel *WKBJ* approximation. A first-order amplitude evolution equation is derived. The solution of the amplitude equation is then used to obtain the first-order approximation in the perturbation field.

Key words: Linear stability, weakly nonlinear theory, method of multiple scales, Ginzburg-Landau equation, collocation method

ANOTĀCIJA

Promocijas darbā tiek veikta plūsmu lineārā un vāji nelineārā stabilitātes analīze seklos sajaukšanās slāņos. Plūsma tiek pieņemta kā nedaudz izliekta garenvirzienā. Lineārā stabilitāte tiek analizēta no laika un telpas aspektiem saskaņā ar „cieta-vāka" pieņēmumu. Atbilstošās lineārās stabilitātes problēmas tiek risinātas skaitliski, izmantojot pseido-spektrālo kolokācijas metodi, kas balstās uz Čebiševa polinomiem. Turklāt problēma ir vispārināta divu komponenšu seklām plūsmām ar lielo Stoksa skaitļu pieņēmumu. Berzes koeficients mainās šķērsvirzienā (literatūrā parasti ir analizēts konstanta berzes koeficienta gadījums, kas ir īpašs gadījums iesniegtā promocijas darbā analīzē).

Ir analizēta bāzes profila asimetrijas ietekme uz stabilitātes parametriem. Tiek izskatītas divas pieejas vāji nelineārās stabilitātes analīzei. Pirmā pieeja pamatojas uz paralēlu plūsmu pieņēmumu. To var izmantot gadījumā, kad gultnes berzes koeficients ir nedaudz mazāks par kritisko vērtību. Izmantojot vairāku mērogu metodi, tiek iegūts amplitūdas evolūcijas vienādojums visvairāk nestabilajam režīmam. Parādīts, ka nedaudz izliektam seklam sajaukšanās slānim, kurš var saturēt vai nesaturēt sīkas daļiņas, amplitūdas vienādojums ir kompleksais Ginzburga-Landau vienādojums. Vienādojuma koeficienti tiek aprēķināti no integrāļiem, kas satur plūsmas lineārās stabilitātes parametrus. Tiek aplūkota plakanu viļņu stabilitāte Ginzburga-Landau vienādojumam. Parādīti Ginzburga-Landau vienādojuma skaitliskie aprēķini dažādām parametru vērtībām un sākuma nosacījumiem.

Otra pieeja ņem vērā lēno garenvirziena bāzes plūsmas izmaiņu. Analīzes pamatā ir vāji neparalēla WKBJ aproksimācija. Tiek iegūts pirmās kārtas amplitūdas attīstības vienādojums. Amplitūdas vienādojuma atrisinājums tiek izmantots, lai iegūtu pirmās kārtas perturbācijas lauka aproksimāciju.

Atslēgas vārdi: Lineārā stabilitāte, vāji nelineārā teorija, vairāku mērogu metode, Ginzburga-Landau vienādojums, kolokācijas metode

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CONTENTS

INTRODUCTION

The Structure of the Thesis

The main goal of the Doctoral Thesis is to develop mathematical models, which can be used to analyse linear and weakly nonlinear instability of shallow mixing layers for the case of a single-component flow or two-component flow. The flow is assumed to be slightly curved along the longitudinal coordinate and the friction coefficient is assumed to be a function of the transverse coordinate. Such a situation describes real flows in compound channels in case of floods.

Chapter 1 (Introduction) presents a review of the literature used in the Doctoral Thesis. Basic equations used in the research are also described.

In Chapter 2, the linear stability and weakly nonlinear methods for analysis of slightly curved shallow mixing layers are presented in detail. Numerical methods used for the solution of stability problems are analysed.

Chapter 3 is devoted to the analysis of a similar problem for the case of slightly curved two-component shallow mixing layers. Linear and weakly nonlinear stability analysis is performed under the assumption of large Stokes numbers.

Chapter 4 is devoted to the spatial stability analysis of slightly curved shallow mixing layers.

Chapter 5 analyses linear and weakly nonlinear instability of shallow mixing layers with variable friction in the transverse direction.

Chapter 6 is devoted to the numerical analysis of solution of Ginzburg-Landau equation.

The Topicality of the Research

The understanding of the interaction between fast and slow fluid streams in shallow mixing layers is important for the analysis of flows at river junctions and for design of compound channels. Real channels and rivers are not straight. Thus, the effect of curvature on the stability characteristics of shallow mixing layers should also be taken into account for proper design and analysis of compound channels. The case of non-uniform friction in the transverse direction is important from an environmental point of view. The friction coefficient in floodplain is usually higher than in the main channel (especially in case of floods).

Complex vortex structures can accumulate contaminants and residues, thereby adversely affecting the environment. Hence, there is a need for a model that describes the shallow flow, as well as methods that allow analysing the flow stability and following up the development of perturbations.

The Objectives of the Doctoral Thesis

1. Analysis of linear and weakly nonlinear stability of slightly curved shallow mixing layers.
2. Investigation of linear and weakly nonlinear stability characteristics of slightly curved two-component shallow mixing layers.
3. Study of spatial stability of slightly curved shallow mixing layers.
4. Investigation of linear and weakly nonlinear instability of shallow mixing layers with variable friction.
5. Numerical analysis of linear and weakly nonlinear models.

Research Methodology

A base flow with a relatively simple structure is selected. Equations of motion are linearized in the neighbourhood of the base flow. The linearized equations are solved by the method of normal modes. The corresponding linear stability problems are solved numerically using pseudo-spectral collocation method based on Chebyshev polynomials.

Two approaches for weakly nonlinear stability analysis of single and two-component slightly curved shallow mixing layers are described. The first approach is based on the parallel flow assumption. Method of multiple scales is used in order to derive an amplitude evolution equation for the most unstable mode. It is shown that the amplitude equation is the complex Ginzburg-Landau equation. The coefficients of the equation are found in closed form in terms of integrals containing the following parameters and functions:

1. Critical values of the bed-friction number, wave number and phase speed of the perturbation.
2. Eigenfunctions of the corresponding adjoint problem.
3. Solutions of three boundary-value problems for ordinary differential equations, one of which is resonantly forced.

4. Solution of the resonantly forced problem is found using singular value decomposition.
5. The other two problems are solved by a collocation method based on Chebyshev polynomials.

The second method takes into account a slow longitudinal variation of the base flow. The analysis is based on weakly non parallel *WKBJ* approximation.

Scientific Novelty and Main Results

- Linear stability problem for slightly curved shallow mixing layers, two-component slightly curved shallow mixing layers and shallow mixing layers with variable friction is formulated and solved numerically for different values of the parameters of the problem.
- Linear stability calculations are performed using temporal and spatial approach.
- It is shown that the amplitude evolution equation under the rigid-lid assumption in a weakly nonlinear regime is the complex Ginzburg-Landau equation.
- Explicit formulas for the computation of the coefficients of the Ginzburg-Landau equation are obtained for slightly curved shallow mixing layers, for slightly curved two-component shallow mixing layers and for shallow mixing layers with variable friction.
- Stability of shallow mixing layers with variable friction in linear and weakly nonlinear case is analysed.
- Amplitude equation describing the evolution of the amplitude of the perturbation with respect to the longitudinal coordinate is derived.
- The derived Ginzburg-Landau equation is solved numerically for different parameters of the problem and different initial conditions.
- Stability of plane wave solutions of the Ginzburg-Landau equation is analysed.

Applications

Understanding stability characteristics and development of instability in shallow flows is important for design of compound channels. Since mixing layers also occur at river junctions and rivers are not straight, the analysis of the effect of curvature should also be taken into account. In some cases, flows can contain heavy particles moving with the fluid. Linear and weakly nonlinear analysis of two-component shallow mixing layers performed in the Thesis explained the effect of particle loading parameter on the stability characteristics of the flow under the assumption of large Stokes numbers.

Shallow water equations are nonlinear. Thus, numerical modelling of shallow water flows requires considerable computational resources since the number of parameters characterising the problem is large. Amplitude evolution equations for problems in thermal convection and Taylor-Couette flows are found to be quite useful in describing the dynamics of the corresponding flows at the initial stages of instability. Amplitude evolution equation in the form of a complex Ginzburg-Landau equation is derived in the Thesis from the equations of motion in a weakly nonlinear regime for the case of single or two-component slightly curved shallow mixing layers where the friction coefficient is constant or non-constant in the transverse direction. Since the Ginzburg-Landau equation is quite rich in terms of different solutions (depending on the values of the coefficients), in many cases it is used as a phenomenological equation for the analysis of spatio-temporal dynamics of complex flows.

The coefficients of the equation are estimated using experimental data, and the equation then can be used to model complex phenomena in fluid mechanics. It is shown in the Thesis that the coefficients of the Ginzburg-Landau equation can be calculated in closed form using linear stability characteristics of the flow. Thus, varying the parameters of the problem and re-calculating the coefficients of the Ginzburg-Landau equation one can use the equation to analyse spatio-temporal dynamics of the flow in a weakly nonlinear regime.

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Presentations at International Conferences

1. Weakly nonlinear analysis of shallow mixing layers with variable friction. *11th World Congress on Computational mechanics, 6th European Conference on Computational Fluid Dynamics*, Spain, Barcelona, 20–25 July 2014.
2. On the stability of shallow mixing layers with non-uniform friction. *The 10th Latvian Mathematical Conference. The 2nd International Conference on High Performance Computing and Mathematical Modelling*, Liepaja, Latvia, 11–12 April 2014.
3. Spatial stability analysis of shallow mixing layers with variable friction coefficient. *IASTED Intern. Conference on Modelling, Identification, and Control, MIC 2014*, Austria, Innsbruck, 17–19 February 2014.
4. Spatial and temporal instability of slightly-curved particle-laden shallow mixing layers. *V Intern. Conference on Computational Methods for Coupled Problems in Science and Eng.*, Spain, Ibiza, 17–19 June 2013.
5. Linear instability of shallow mixing layers with non-constant friction coefficient. *International Conference on Applied Mathematics and Scientific Computing*, Croatia, Šibenik, 10–14 June 2013.
6. Spatial stability analysis of curved shallow mixing layers. *15th International Conference on Mathematical Methods, Computational Techniques and Intelligent Systems*, Cyprus, Limassol, 21–23 March 2013.
7. Spatial instability of curved shallow mixing layers. *17th Intern. Conf. on Mathematical Modelling and Analysis,* Estonia, Tallinn, 6–9 June 2012.
8. Weakly nonlinear methods for stability analysis of slightly curved two-phase shallow mixing layers. *International Conference on Applied Mathematics and Sustainable Development: Special Track within SCET2012*, China, Xi'an, 27–30 May 2012.
9. Ginzburg-Landau model for curved two-phase shallow mixing layers. *ICCAM 2012: International Conference on Computational and Applied Mathematics*, Italy, Venice, 11–13 April 2012.
10. Asymptotic analysis of stability of slightly curved two-phase shallow mixing layers. *International Conference on Fluid Mechanics and Heat & Mass Transfer*, Greece, Corfu, 14–16 July 2011.
11. The effect of flow curvature on linear and weakly nonlinear instability of shallow mixing layers. *16th International Conference on Mathematical Modelling and Analysis*, Latvia, Sigulda, 25–28 May 2011.

12. The effect of slow variation of base flow profile on the stability of slightly curved mixing layers. *WASET International Conference*. Italy, Venice 27-29 April 2011.
13. Linear instability of curved shallow mixing layers. *The 6th IASME / WSEAS International Conference on Continuum Mechanics (CM'11)*, United Kingdom, Cambridge, 23–25 February 2011.
14. Spatial instability of asymmetric base flow profiles in shallow water. *15th International Conference Mathematical Modelling and Analysis*, Lithuania, Druskininkai, 26–29 May 2010.
15. The effect of asymmetry of base flow profile on the linear stability of shallow mixing layers. *10th WSEAS Intern. Conf. on Wavelet Analysis and Multirate Systems*, Tunisia, Kantaoui, Sousse, 3–6 May 2010.

Presentations at Local Conferences

1. Amplitūdas evolūcijas vienādojums stabilitātes analīzei divu fāžu sekla ūdens plūsmām. *9. Latvijas matemātikas konference.* Latvia, Jelgava, 30-31 March 2012.
2. Sekla ūdens plūsmas stabilitāte gadījumā, ja bāzes plūsmas profils nav simetrisks. 8. *Latvijas matemātikas konference.* Latvia, Valmiera, 9–10 April 2010.

1. MATHEMATICAL FORMULATION OF THE PROBLEM

1.1 Literature Survey

Linear stability theory is widely used in order to analyse the behaviour of fluid flows (see, for example, [9], [11], [52] and [60]). In many engineering applications of fluid mechanics the transverse length scale of the flow is much larger than water depth. Such flows are usually referred to as “shallow flows”. Curved shallow mixing layers are of a particular interest (flows in compound and composite channels and flows at river junctions represent typical examples of shallow mixing layers). Methods of analysis of shallow mixing layers include experimental investigation, numerical modelling and stability analysis [41]. Experimental investigation of shallow mixing layers is conducted in many papers (see, for example, [6], [64] and [65]). It is shown in these papers that bottom friction plays an important role in suppressing perturbations. In addition, the rate of growth of the mixing layer is also reduced in comparison with the case of free mixing layers.

Linear stability analysis of shallow flows is performed in [5], [7], [33], [43], [46] and [57]. Rigid-lid assumption is used in [7] to determine the critical values of the bed friction number for wake flows and mixing layers. The applicability of the rigid-lid assumption to the stability analyses of shallow flows is analysed in [33], where it is shown that for small Froude numbers the error in using the rigid-lid assumption is quite small. The effect of Froude number of the stability of shallow mixing layers in compound and composite channels is studied in [43]. Theoretical results and numerical computations presented in [5], [7], [33], [43], [46] and [57] confirm experimental observations: the bed friction number stabilizes the flow and reduces the growth of a mixing layer.

Centrifugal instability can also occur in shallow mixing layers. The effect of small curvature of the stability of free mixing layers is investigated in [35], [40] and [53]. It is shown in [53] that curvature has a stabilizing effect on a stably curved mixing layer and destabilizing effect on unstably curved mixing layer.

Linear stability analysis can be used to determine how a particular flow becomes unstable. Critical values of the parameters (for example, critical bed friction number, critical wave number and so on) are also estimated from the linear stability theory. Development of instability above the threshold cannot be analysed by linear theory. Weakly nonlinear theories [36], [62] are used in order to construct an amplitude evolution equation for the most unstable mode. These theories are based on the method of multiple scales [42] and are applicable if the flow is unstable but the value of the parameter (for example, Reynolds number for channel flows or bed friction number for shallow flows) is close to the critical value. In this case the growth rate of unstable perturbation is small and one can hope to analyse the development of instability by means of relatively simple evolution equations. Such an approach is used in [62] for plane Poiseuille flow, in [2] and [49] in order to analyse instability of waves generated by wind and in [30], [33], [45], [46] and [55] for shallow wake flows. In fact, amplitude equations are used in the literature in two ways. First, a particular form of the evolution equation is selected a priori and the coefficients of the equation are estimated from experimental data. Then the equation with estimated coefficients is used to model the phenomenon of interest. Second, one can actually derive an evolution equation from the equations of motion. This approach is used in [46], [62], [2], [30] and [47] where it is shown that for two-dimensional cases the evolution equation is the complex Ginzbrug-Landau equation.

Ginzburg-Landau equation is often used to model spatio-temporal dynamics of complex flows. In many cases the Ginzburg-Landau equation is used as a phenomenological model, that is, it is assumed but not derived from the equations of motion. Experimental data are often used in such cases in order to estimate the coefficients of the equation.

In other cases the Ginzburg-Landau equation can be derived from the equations of motion (examples are given in [50], [58] and [61]). The coefficients of the equation are calculated in a closed form as integrals containing characteristics of the linearized problems.

Ginzburg-Landau equation and its properties are extensively studied in the literature (see, for example, [1] and [10]). Numerical analysis of the Ginzburg-Landau equation is simpler than numerical solution of the equations of motion. In addition, analysis of stability of some simple (for example, periodic) solutions of the Ginzburg-Landau equation allows researchers to simplify the analysis of spatio-temporal dynamics of complex flows in fluid mechanics.

Linear instability of shallow mixing layers is analysed in [4], [7], [33], [43] under the assumption that bottom friction is modelled by means of the Chezy formula [51] where the friction coefficient is assumed to be constant. Usually the friction coefficient is obtained from semi-empirical formulas [59] which relate the value of the friction coefficient to the Reynolds number of the flow and roughness of the surface. In such a case the friction coefficient is assumed to be constant in the whole region of the flow.

In some applications friction varies considerably in the transverse direction. One particular example is related to shallow flows under condition of partial vegetation. This situation often occurs during floods [66]. Friction force in a partially vegetated area is larger than in the main channel. It is shown in this case that the base flow profile is distorted and becomes asymmetric [66]. The difference in friction forces between partially vegetated area and the main channel is modelled in [66] by a step function. Linear stability analysis is conducted in [66] under the assumption that the base flow profile is symmetric.

1.2 Shallow Water Equations

Shallow water equations are depth-averaged equations which are obtained by integrating equations of fluid mechanics with respect to the vertical coordinate. Since integration takes place over water depth it is necessary to specify stresses at the free surface and at the bottom. Stresses at the free surface are usually much smaller than the stresses at the bottom so that only bottom stresses are usually taken into account in shallow water equations. Empirical formulas (such as Chezy or Manning formulas) are used in practice in order to represent bottom friction. The detailed derivation of shallow water equations is given in [4].

Shallow water equations under the rigid-lid assumption in the presence of a small curvature have the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_{\mathrm{f}}}{2h} u\sqrt{u^2+v^2} - B(u^{\mathrm{p}} - u) = 0, \tag{1.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{1}{R}u^2 + \frac{\partial p}{\partial y} + \frac{c_{\mathrm{f}}}{2h} v\sqrt{u^2+v^2} - B(v^{\mathrm{p}} - v) = 0\,, \tag{1.3}$$

where x, y – geometric coordinates;

t – time;

u and v – the depth-averaged velocity components in the x and y directions;

p – the pressure;

h – water depth;

c_{f} – friction coefficient (can be constant or function of y);

B – particle loading parameter (see [67], [68]);

u^{p} and v^{p} – the components of particle velocities;

$\frac{1}{R} = \frac{\delta_*}{R_*} << 1$ – small parameter;

R_* – the radius of curvature of the centreline of the curved mixing layer;

δ_* – the thickness of the mixing layer.

It is assumed in (1.2), (1.3) that the flow can contain heavy particles. The "lumped" effect of the particles is represented by the particle loading parameter B. Equations (1.1)-(1.3) are written under the assumption of large Stokes number which implies that there is no dynamic interaction between the particles and the carrier fluid.

Water surface in (1.1)-(1.3) is treated as the "rigid-lid" (in other words, water depth is assumed to be constant). Bottom friction in (1.2), (1.3) is modelled by means of the Chezy formula (see [4]).

As it is shown in [33], the rigid-lid assumption (from a linear stability point of view) is valid for small Froude numbers.

Following [67], [68] we assume that the following conditions are satisfied with respect to the distribution of particles within a carrier fluid:

1. The particles are spheres with small diameters.
2. The diameters are small in comparison with the dimensions of large-scale structures.
3. The particles and the flow are in a dynamic equilibrium at the beginning of the transient.
4. The material density of the particle is much larger than that of the fluid.
5. The small perturbations imposed on the flow have no effect on the particles during the initial moment.

Friction coefficient c_f in some applications varies in the transverse direction. Examples include shallow flows under conditions of partial vegetation during floods where water flows through partially vegetated area [67] or flows in compound and composite channels [43]. In such cases the resistance force in the main channel is usually smaller than in the vegetated area of a composite channel or in the shallower area of a compound channel. The variability of the friction coefficient in the transverse direction is modelled by a smooth differentiable shape function $c_f(y)$.

2. STABILITY OF SLIGHTLY CURVED SHALLOW MIXING LAYERS

2.1 Linear Stability

Consider shallow water equations under the rigid-lid assumption in the presence of a small curvature in the form (1.1)–(1.3), where $B = 0$ ([15], [16], [17], [26]).

Eliminating the pressure p (differentiating (1.2) with respect to y and (1.3) with respect to x) we obtain:

$$\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 p}{\partial x\partial y}+\frac{c_{\mathrm{f}}}{2h}\frac{\partial u}{\partial y}\sqrt{u^2+v^2}+\frac{c_{\mathrm{f}}}{2h}u\frac{u\dfrac{\partial u}{\partial y}+v\dfrac{\partial v}{\partial y}}{\sqrt{u^2+v^2}}=0$$

$$\frac{\partial^2 v}{\partial t\partial x}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 p}{\partial y\partial x}+\frac{c_{\mathrm{f}}}{2h}\frac{\partial v}{\partial x}\sqrt{u^2+v^2}-\frac{2u}{R}\frac{\partial u}{\partial x}+\frac{c_{\mathrm{f}}}{2h}v\frac{u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial x}}{\sqrt{u^2+v^2}}=0$$

Subtracting the second equation from the first we obtain:

$$\frac{\partial^2 u}{\partial t\partial y}-\frac{\partial^2 v}{\partial t\partial x}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}-u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 u}{\partial y^2}-v\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 p}{\partial x\partial y}$$

$$-\frac{\partial^2 p}{\partial y\partial x}+\frac{c_{\mathrm{f}}}{2h}\frac{\partial u}{\partial y}\sqrt{u^2+v^2}-\frac{c_{\mathrm{f}}}{2h}\frac{\partial v}{\partial x}\sqrt{u^2+v^2}+\frac{c_{\mathrm{f}}}{2h}u\frac{u\dfrac{\partial u}{\partial y}+v\dfrac{\partial v}{\partial y}}{\sqrt{u^2+v^2}}-\frac{c_{\mathrm{f}}}{2h}v\frac{u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial x}}{\sqrt{u^2+v^2}}+\frac{2u}{R}\frac{\partial u}{\partial x}=0$$

Simplifying the resulting equation we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{\partial v}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)$$

$$+\frac{2u}{R}\frac{\partial u}{\partial x}+\frac{c_{\mathrm{f}}}{2h}\sqrt{u^2+v^2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{c_{\mathrm{f}}}{2h\sqrt{u^2+v^2}}\left(u^2\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial y}-uv\frac{\partial u}{\partial x}-v^2\frac{\partial v}{\partial x}\right)=0$$

Introducing the stream function $\psi(x,y,t)$ by the relations

$$u=\frac{\partial\psi}{\partial y}=\psi_y,\ v=-\frac{\partial\psi}{\partial x}=-\psi_x \tag{2.1}$$

and using the notation $\Delta\psi=\psi_{xx}+\psi_{yy}$ we rewrite (1.1)–(1.3) in the form

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}$$
$$+\frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) = 0 \quad , \qquad (2.2)$$

where the subscripts indicate the derivatives with respect to the variables x, y and t.

Here the parallel flow assumption is used. Experiments [64], [65] show that the base flow slightly changes downstream. The parallel flow assumption implies that the base flow does not change in the longitudinal direction. As pointed out in [50] this approximation is the leading-order solution in a multiple-scale expansion which takes into account slow flow divergence.

Consider the stream function $\psi(x,y,t)$ of the form

$$\psi = \psi_0 + \psi', \qquad (2.3)$$

where the quantity with prime represent small perturbations.

Substituting (2.3) into (2.2) we obtain the following equation:

$$(\Delta(\psi_0+\psi'))_t + (\psi_0+\psi')_y(\Delta(\psi_0+\psi'))_x - (\psi_0+\psi')_x(\Delta(\psi_0+\psi'))_y$$
$$+\frac{2}{R}(\psi_0+\psi')_y(\psi_0+\psi')_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta(\psi_0+\psi')\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2}$$
$$+\frac{c_{\mathrm{f}}}{2h\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2}}\begin{pmatrix}(\psi_0+\psi')_y^2(\psi_0+\psi')_{yy}\\ +2(\psi_0+\psi')_x(\psi_0+\psi')_y(\psi_0+\psi')_{xy}\\ +(\psi_0+\psi')_x^2(\psi_0+\psi')_{xx}\end{pmatrix} = 0.$$

We assume that perturbations are small so that quadratic or higher terms in the equations may be ignored. Using the Maclaurin series $\sqrt{1+x} = 1+\frac{x}{2}+...$ we rewrite the expression $\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2}$ in the form:

$$\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2} \Rightarrow \sqrt{\psi_{oy}^2+2\psi_{0y}\psi_y'} = \psi_{0y}\sqrt{1+\frac{2\psi_y'}{\psi_{0y}}} = \psi_{0y}\left(1+\frac{\psi_y'}{\psi_{0y}}\right) = \psi_{0y}+\psi'.$$

Thus,

$$\frac{c_{\mathrm{f}}}{2h}\Delta(\psi_0+\psi')\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2}\Rightarrow\frac{c_{\mathrm{f}}}{2h}(\psi_{0yy}+\psi'_{yy}+\psi'_{xx})(\psi_{0y}+\psi'_y)$$

$$\Rightarrow\frac{c_{\mathrm{f}}}{2h}(\psi_{0yy}\psi_{0y}+\psi'_{yy}\psi_{0y}+\psi'_{xx}\psi_{0y}+\psi_{0yy}\psi'_y)$$

$$\frac{c_{\mathrm{f}}}{2h\sqrt{(\psi_0+\psi')_x^2+(\psi_0+\psi')_y^2}}\begin{pmatrix}(\psi_0+\psi')_y^2(\psi_0+\psi')_{yy}\\+2(\psi_0+\psi')_x(\psi_0+\psi')_y(\psi_0+\psi')_{xy}\\+(\psi_0+\psi')_x^2(\psi_0+\psi')_{xx}\end{pmatrix}$$

$$\Rightarrow\frac{c_{\mathrm{f}}}{2h}(\psi_{0yy}\psi_{0y}+\psi'_{yy}\psi_{0y}+\psi_{0yy}\psi'_y)$$

Dropping the primes we obtain the following equation

$$\begin{aligned}&\psi_{xxt}+\psi_{yyt}+\psi_{0y}(\psi_{xxx}+\psi_{xyy})-\psi_{0yyy}\psi_x\\&+\frac{c_{\mathrm{f}}}{2h}(\psi_{0y}\psi_{xx}+2\psi_{0yy}\psi_y+2\psi_{0y}\psi_{yy})+\frac{2}{R}\psi_{0y}\psi_{xy}=0\end{aligned}. \tag{2.4}$$

Following the method of normal modes [11] we assume a perturbation of the form

$$\psi(x,y,t)=\varphi(y)\mathrm{e}^{ik(x-ct)}, \tag{2.5}$$

where $\varphi(y)$ – the amplitude of the normal perturbation;

k – the wave number;

c – the phase speed of the perturbation.

The derivatives of ψ with respect to x, y or t are

$$\begin{aligned}&\psi_x=\varphi(y)\mathrm{e}^{ik(x-ct)}ik; &&\psi_{xx}=-\varphi(y)\mathrm{e}^{ik(x-ct)}k^2;\\&\psi_{xxx}=-\varphi(y)\mathrm{e}^{ik(x-ct)}ik^3; &&\psi_{xy}=\varphi'(y)\mathrm{e}^{ik(x-ct)}ik;\\&\psi_{xxx}=-\varphi(y)\mathrm{e}^{ik(x-ct)}ik^3; &&\psi_{xy}=\varphi'(y)\mathrm{e}^{ik(x-ct)}ik;\\&\psi_{xyy}=\varphi''(y)\mathrm{e}^{ik(x-ct)}ik; &&\psi_{xxt}=\varphi(y)\mathrm{e}^{ik(x-ct)}ik^3c;\\&\psi_y=\varphi'(y)\mathrm{e}^{ik(x-ct)}; &&\psi_{yy}=\varphi''(y)\mathrm{e}^{ik(x-ct)};\\&\psi_{yyt}=-\varphi''(y)\mathrm{e}^{ik(x-ct)}ikc; &&\psi_{xy}=\varphi'(y)\mathrm{e}^{ik(x-ct)}ik.\end{aligned} \tag{2.6}$$

Substituting (2.5) and derivatives (2.6) into (2.4) and using the notation

$$U=\psi_{0y}, \tag{2.7}$$

we obtain:

$$-ik^3c\varphi - ikc\varphi'' + U\left(-ik^3 + ik\varphi''\right) - ikU_{yy}\varphi + \frac{c_\mathrm{f}}{2h}\left(k^2U + 2U_y\varphi' + 2U\varphi''\right) + \frac{2}{R}ikU\varphi' = 0,$$

or

$$\varphi''\left(ik(U-c) + SU\right) + \varphi'\left(\frac{2}{R}ikU + SU_y\right) + \varphi\left(ik^3c - ik^3U - ikU_{yy} - \frac{k^2SU}{2}\right) = 0, \quad (2.8)$$

where $S = \dfrac{c_\mathrm{f}\delta_*}{h}$ – the bed-friction number;

δ_* – the width of the mixing layer.

Dividing the equation by *ik* we obtain:

$$\varphi''\left(U - c - \frac{iSU}{k}\right) + \varphi'\left(\frac{2}{R}U - \frac{iSU_y}{k}\right) + \varphi\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2}\right) = 0. \quad (2.9)$$

The boundary conditions are

$$\varphi(\pm\infty) = 0. \quad (2.10)$$

Using linear stability theory one can determine the conditions under which a particular flow becomes unstable. The eigenvalues $c = c_\mathrm{r} + ic_\mathrm{i}$ determine the linear stability of base flow. The base flow is said to be linearly stable if all $c_\mathrm{i} < 0$, and unstable, if at least one $c_\mathrm{i} > 0$. Numerical solution of the corresponding eigenvalue problem (2.10) – (2.9) allows one to obtain the critical values of the parameters of the problem and determine the structure of the unstable mode. However, linear theory cannot be used to predict the evolution of the most unstable mode above the threshold. In the unstable region perturbation grows exponentially with time (see (2.5)). If the growth rate is large then nonlinear effects quickly become dominant and there is little hope to analyse the development of instability analytically. However, if the growth rate of the unstable mode is relatively small then weakly nonlinear theories can be used in order to develop an amplitude evolution equation for the most unstable mode.

In a classical theory of hydrodynamic stability [11] the base flow is usually a simple solution of the equations of motion. As an example we consider the Navier-Stokes equations where the velocity vector has only one nonzero component which is a function of a radial coordinate only. Solving the Navier-Stokes equations we obtain a parabolic velocity distribution (the Poiseuille flow). This approach does not work for shallow water equations: it is not possible to find a simple analytical solution $U(y)$ of (1.1) – (1.3). Base flows in the case of shallow water equations are usually chosen in the form of relatively simple model velocity profiles such as hyperbolic tangent profile for shallow mixing layers or hyperbolic secant profile for shallow wake flows. These profiles are chosen on the basis of careful analysis of available experimental data. The following two base flow profiles will be used below:

$$U(y) = 2 + \tanh y \tag{2.11}$$

and

$$U(y) = 2 - \tanh y. \tag{2.12}$$

a) b)

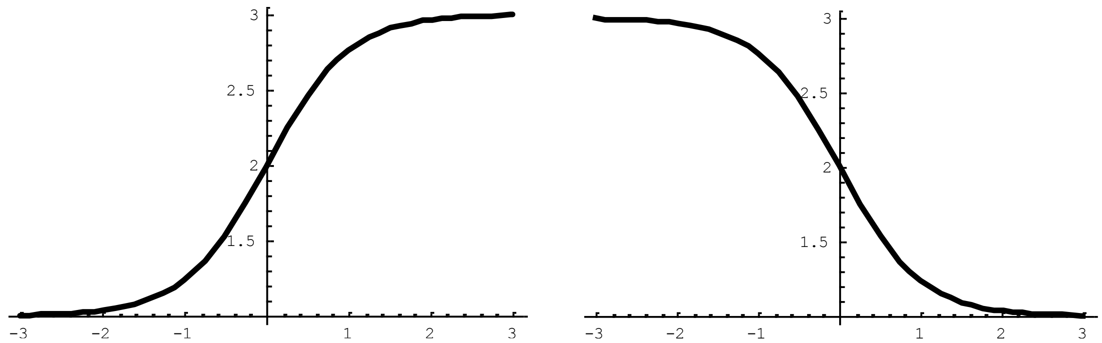

Fig. 2.1. Base flow profile a) $U(y) = 2 + \tanh y$ and b) $U(y) = 2 - \tanh y$

Velocity profile (2.11) – Fig.2.1 a) corresponds to stably curved mixing layer (in this case the high-speed stream is on the outside of the low-speed stream). Profile (2.12) – Fig.2.1 b) represents the opposite situation (the high-speed stream is on the inside of the low-speed stream). It is shown in [15] that experimentally observed base flow velocity profile has similar shape to that of the plane mixing layer.

2.2 Numerical Method for Linear Stability

In this subsection we describe a numerical method for the calculation of the marginal stability curves and growth rates of unsteady perturbations.

The pseudospectral collocation method based on Chebyshev polynomials is used to solve eigenvalue problem (2.9)-(2.10) numerically. The interval $-\infty < y < +\infty$ is transformed into the interval $(-1, 1)$ by means of the transformation $r = \dfrac{2}{\pi}\arctan y$. The solution to (2.9) is then sought in the form

$$\varphi(r) = \sum_{j=0}^{N-1} a_j (1-r^2) T_j(r), \tag{2.13}$$

where $T_j(r) = \cos j \arccos r$ – the Chebyshev polynomial of the first kind of degree j;

a_j – unknown coefficients.

The factor $1-r^2$ guarantees that the boundary conditions (2.10) in terms of the new variable r are satisfied automatically at $r = \pm 1$. The use of the base functions that satisfy the given zero boundary conditions considerably reduces the condition number of the matrix obtained after discretization [38].

Using the chain rule we compute the derivatives of the first and second order of φ with respect to y:

$$\frac{d\varphi}{dy} = \frac{d\varphi}{dr}\frac{dr}{dy} = \frac{2}{\pi(1+y^2)}\frac{d\varphi}{dr} = \left| 1+y^2 = 1+\tan^2\frac{\pi r}{2} = \frac{1}{\cos^2\frac{\pi r}{2}} \right| = \frac{2}{\pi}\cos^2\frac{\pi r}{2}\frac{d\varphi}{dr},$$

$$\begin{aligned}\frac{d^2\varphi}{dy^2} &= \frac{d}{dy}\left(\frac{2}{\pi}\cos^2\frac{\pi r}{2}\frac{d\varphi}{dr}\right) = -\frac{4}{\pi(1+y^2)^2}\frac{d\varphi}{dr} + \frac{4}{\pi^2(1+y^2)^2}\frac{d^2\varphi}{dr^2} \\ &= \frac{4}{\pi^2}\cos^4\frac{\pi r}{2}\frac{d^2\varphi}{dr^2} - \frac{4}{\pi}\sin\frac{\pi r}{2}\cos^3\frac{\pi r}{2}\frac{d\varphi}{dr}.\end{aligned} \tag{2.14}$$

The derivatives of φ with respect to r are evaluated using (2.13):

$$\begin{aligned}\frac{d\varphi}{dr} &= \sum_{j=0}^{N-1} a_j[-2rT_j(r) + (1-r^2)T_j'(r)], \\ \frac{d^2\varphi}{dr^2} &= \sum_{j=0}^{N-1} a_j[-2T_j(r) - 4rT_j'(r) + (1-r^2)T_j''(r)].\end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.14) the derivatives of the first and second order of φ with respect to y will be as follows:

$$\frac{d\varphi}{dy} = \frac{2}{\pi}\cos^2\frac{\pi r}{2}\sum_{j=0}^{N-1} a_j\left(-2rT_j(r) + (1-r^2)T_j^{'}(r)\right),$$

$$\frac{d^2\varphi}{dy^2} = \frac{4}{\pi^2}\cos^4\frac{\pi r}{2}\sum_{j=0}^{N-1} a_j\left(-2T_j(r) - 4rT_j^{'}(r) + (1-r^2)T_j^{''}(r)\right) \tag{2.16}$$
$$-\frac{4}{\pi}\sin\frac{\pi r}{2}\cos^3\frac{\pi r}{2}\sum_{j=0}^{N-1} a_j\left(-2rT_j(r) + (1-r^2)T_j^{'}(r)\right).$$

The following set of collocation points is used to solve (2.9), (2.10):

$$r_m = \cos\frac{\pi m}{N+1}, \quad m = 1,2,..., N. \tag{2.17}$$

In order to evaluate the function $\varphi(r)$ and its derivatives up to the second order we need to compute the values of the Chebyshev polynomial $T_j(r)$ and its derivatives at the collocation points (2.17):

$$T_j(r_m) = \cos j\arccos(\cos\frac{\pi m}{N+1}) = \cos\frac{j\pi m}{N+1},$$

$$T_j^{'}(r_m) = \frac{j}{\sqrt{1-r_m^2}}\sin j\arccos(\cos\frac{\pi m}{N+1}) = \frac{j\sin\frac{j\pi m}{N+1}}{\sin\frac{\pi m}{N+1}}, \tag{2.18}$$

$$T_j^{''}(r_m) = \frac{j\cos\frac{\pi m}{N+1}\sin\frac{j\pi m}{N+1}}{\sin^3\frac{\pi m}{N+1}} - \frac{j^2\cos\frac{j\pi m}{N+1}}{\sin^2\frac{\pi m}{N+1}}.$$

Substituting (2.13-2.18) into (2.9) we obtain the linear system of the equations ($m = 1,2,..., N$) in the form:

$$\left(U-c-\frac{iSU}{k}\right)\left(\begin{array}{l}\dfrac{4}{\pi^2}\cos^4\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(\begin{array}{l}-2\cos\dfrac{j\pi m}{N+1}-4\cos\dfrac{\pi m}{N+1}\cdot\dfrac{j\sin\frac{j\pi m}{N+1}}{\sin\frac{\pi m}{N+1}}\\ +\sin^2\dfrac{\pi m}{N+1}\left(\begin{array}{l}\dfrac{j\cos\frac{\pi m}{N+1}\sin\frac{j\pi m}{N+1}}{\sin^3\frac{\pi m}{N+1}}\\ -\dfrac{j^2\cos\frac{j\pi m}{N+1}}{\sin^2\frac{\pi m}{N+1}}\end{array}\right)\end{array}\right)\\ -\dfrac{4}{\pi}\sin\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\cos^3\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(\begin{array}{l}-2\cos\dfrac{\pi m}{N+1}\cos\dfrac{j\pi m}{N+1}\\ +\sin^2\dfrac{\pi m}{N+1}\cdot\dfrac{j\sin\frac{j\pi m}{N+1}}{\sin\frac{\pi m}{N+1}}\end{array}\right)\end{array}\right)$$

$$+\left(\frac{2}{R}U-\frac{iSU_y}{k}\right)\frac{2}{\pi}\cos^2\frac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(-2\cos\frac{\pi m}{N+1}\cos\frac{j\pi m}{N+1}+\sin^2\frac{\pi m}{N+1}\cdot\frac{j\sin\frac{j\pi m}{N+1}}{\sin\frac{\pi m}{N+1}}\right)$$

$$+\left(k^2c-k^2U-U_{yy}+\frac{ikSU}{2}\right)\sum_{j=0}^{N-1}a_j\sin^2\frac{\pi m}{N+1}\cos\frac{j\pi m}{N+1}=0.$$

Simplifying the obtained equation we get

$$\left(U-c-\frac{iSU}{k}\right)\left(\begin{array}{l}\dfrac{4}{\pi^2}\cos^4\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(-2\cos\dfrac{j\pi m}{N+1}-3j\sin\dfrac{j\pi m}{N+1}\cot\dfrac{\pi m}{N+1}-j^2\cos\dfrac{j\pi m}{N+1}\right)\\ -\dfrac{4}{\pi}\sin\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\cos^3\dfrac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(\begin{array}{l}-2\cos\dfrac{j\pi m}{N+1}\cos\dfrac{\pi m}{N+1}\\ +j\sin\dfrac{j\pi m}{N+1}\sin\dfrac{\pi m}{N+1}\end{array}\right)\end{array}\right)$$

$$+\left(\frac{2}{R}U-\frac{iSU_y}{k}\right)\frac{2}{\pi}\cos^2\frac{\pi\cos\frac{\pi m}{N+1}}{2}\sum_{j=0}^{N-1}a_j\left(-2\cos\frac{j\pi m}{N+1}\cdot\cos\frac{\pi m}{N+1}+j\cdot\sin\frac{j\pi m}{N+1}\cdot\sin\frac{\pi m}{N+1}\right)$$

$$+\left(k^2c-k^2U-U_{yy}+\frac{ikSU}{2}\right)\sum_{j=0}^{N-1}a_j\cos\frac{j\pi m}{N+1}\sin^2\frac{\pi m}{N+1}=0.$$

Reshuffle terms which contain the factor c we obtain the linear system of the equations ($m=1,2,...,N$) in the form:

$$\sum_{j=0}^{N-1} a_j \left(\begin{array}{l} \left(U - \dfrac{iSU}{k}\right) \left(\begin{array}{l} \dfrac{4}{\pi^2} \cos^4 \dfrac{\pi \cos \dfrac{\pi m}{N+1}}{2} \left(-2\cos\dfrac{j\pi m}{N+1} - 3j\sin\dfrac{j\pi m}{N+1}\cot\dfrac{\pi m}{N+1} - j^2\cos\dfrac{j\pi m}{N+1} \right) \\ -\dfrac{4}{\pi}\sin\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2}\cos^3\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2} \left(\begin{array}{l} -2\cos\dfrac{j\pi m}{N+1}\cdot\cos\dfrac{\pi m}{N+1} \\ +j\cdot\sin\dfrac{j\pi m}{N+1}\cdot\sin\dfrac{\pi m}{N+1} \end{array} \right) \end{array} \right) \\ \left(\dfrac{2}{R}U - \dfrac{iSU_y}{k}\right)\dfrac{2}{\pi}\cos^2\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2}\left(-2\cos\dfrac{j\pi m}{N+1}\cdot\cos\dfrac{\pi m}{N+1} + j\cdot\sin\dfrac{j\pi m}{N+1}\cdot\sin\dfrac{\pi m}{N+1}\right) \\ -\left(k^2U + U_{yy} - \dfrac{ikSU}{2}\right)\cos\dfrac{j\pi m}{N+1}\cdot\sin^2\dfrac{\pi m}{N+1} \end{array} \right)$$

$$-c\cdot\sum_{j=0}^{N-1} a_j \left(\begin{array}{l} \dfrac{4}{\pi^2}\cos^4\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2}\left(-2\cos\dfrac{j\pi m}{N+1} - 3j\sin\dfrac{j\pi m}{N+1}\cot\dfrac{\pi m}{N+1} - j^2\cos\dfrac{j\pi m}{N+1}\right) \\ -\dfrac{4}{\pi}\sin\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2}\cos^3\dfrac{\pi\cos\dfrac{\pi m}{N+1}}{2}\left(\begin{array}{l} -2\cos\dfrac{j\pi m}{N+1}\cos\dfrac{\pi m}{N+1} \\ +j\sin\dfrac{j\pi m}{N+1}\sin\dfrac{\pi m}{N+1} \\ -k^2\cos\dfrac{j\pi m}{N+1}\sin^2\dfrac{\pi m}{N+1} \end{array} \right) \end{array} \right) = 0$$

or

$$\sum_{j=0}^{N-1} a_j \big(B(j,m)\big) - c\sum_{j=0}^{N-1} a_j \big(D(j,m)\big) = 0\ .$$

We obtain the generalized eigenvalue problem of the form

$$(B - cD)a = 0, \qquad (2.19)$$

where B and D are complex-values $N \times N$ matrices and $a = (a_0\, a_1 ... a_{N-1})^{\mathrm{T}}$.

There are at least two reasons why solutions of the form (2.13) are more convenient than those obtained by “classical” collocation methods [3]:

1. The use of the base functions that satisfy the given zero boundary conditions considerably reduces the condition number [39].
2. The matrix D in (2.19) is not singular.

Problem (2.19) is solved numerically by means of the *IMSL* (International Mathematics and Statistics Library) routine *DGVCCG* (Computes all of the eigenvalues and eigenvectors of a generalized complex eigensystem $Az = \lambda Bz$).

The results of numerical computations for the case of stably curved shallow mixing layer (base flow velocity profile (2.11) – Fig. 2.1.a) are shown in Table 2.1.

Table 2.1. The Results of Numerical Computations for the Case of Stably Curved Shallow Mixing Layer (Base Flow Velocity Profile (2.11)).

k	*S*(1/*R*=0)	*S*(1/*R*=0,01)	*S*(1/*R*=0.02)	*S*(1/*R*=0.03)	*S*(1/*R*=0.04)
0.1	0.0260	0.0230	0.0205	0.0194	0.0258
0.2	0.0441	0.0408	0.0377	0.0348	0.0321
0.3	0.0554	0.0519	0.0485	0.0452	0.0421
0.4	0.0609	0.0572	0.0536	0.0501	0.0466
0.5	0.0612	0.0574	0.0536	0.0499	0.0462
0.6	0.0568	0.0529	0.0490	0.0451	0.0412
0.7	0.0482	0.0442	0.0402	0.0361	0.0322
0.8	0.0357	0.0316	0.0275	0.0234	0.0224
0.9	0.0196	0.0154	0.0150	0.0142	0.0138

The results of numerical computations for the case of stably curved shallow mixing layer (base flow velocity profile (2.11)) are shown in Fig. 2.2. Three marginal stability curves are shown in Fig. 2.2 for the three values of the parameter $1/R$, namely, $1/R = 0$, 0.02 and 0.04, respectively (from top to bottom). The region of instability is below the curves [16].

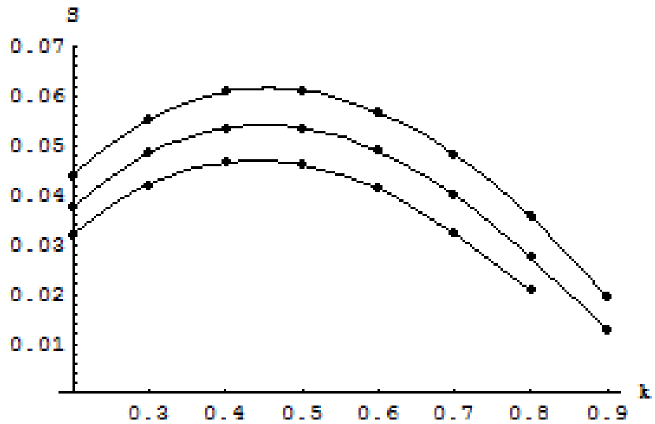

Fig. 2.2. Marginal stability curves for base flow profile (2.11)
The values of the parameter $1/R$ are 0, 0.02 and 0.04, respectively (from top to bottom).

As can be seen from Fig. 2.2 curvature has a stabilizing influence on stably curved shallow mixing layer: the critical values of the bed-friction number S decrease as the parameter $1/R$ increases.

Marginal stability curves for unstably curved shallow mixing layer (base flow profile (2.12)) are shown in Fig. 2.3.

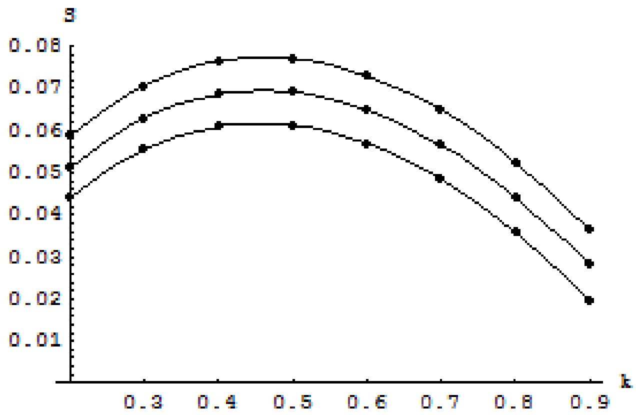

Fig.2.3. Marginal stability curves for base flow profile (2.12)

The values of the parameter $1/R$ are 0.04, 0.02 and 0, respectively (from top to bottom)

The results shown in Fig. 2.3 indicate that the increase of the parameter $1/R$ has a destabilizing influence on unstably curved base flow profile (2.12): the critical values of the bed-friction number increase for larger $1/R$.

Results of numerical computations show that the curvature stabilizes the flow in the case of stably curved mixing layer while for unstably curved mixing layer the curvature has a destabilizing effect on the flow.

2.3 Weakly Nonlinear Methods for Analysis of Shallow Flows

Weakly nonlinear theories are usually constructed in the neighbourhood of a critical point (see Fig. 2.4). Such equations are obtained in the past for the case of plane Poiseuille flow, shallow water flows, waves on the surface generated by wind and in some other situations (see [2], [30], [34], [46], [47], [49], [62]).

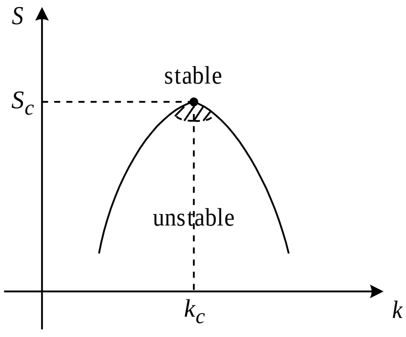

Fig. 2.4. A typical marginal stability curve for shallow water flow.

Suppose that $S_{\rm c}, k_{\rm c}$ and $c_{\rm c}$ are the critical values of the stability parameter, wave number and wave speed, respectively. Then the most unstable mode (in accordance with the linear theory) is given by (2.5) with $S = S_{\rm c}, k = k_{\rm c}$ and $c = c_{\rm c}$ where the eigenfunction $\varphi(y)$ can be replaced by $C\varphi(y)$. The constant C cannot be determined from the linear stability theory. In order to analyse the development of instability analytically in the framework of weakly nonlinear theory we consider a small neighbourhood of the critical point in the (k, S)-plane where parameter S is assumed to be slightly below the critical value:

$$S = S_{\rm c}(1-\varepsilon^2). \qquad (2.20)$$

The constant C in this case will be replaced by a slowly varying amplitude function A. Following the paper by Stewartson and Stuart [62] we introduce the “slow” time τ and longitudinal coordinates ξ by the relations

$$\tau = \varepsilon^2 t,\ \ \xi = \varepsilon(x - c_g t), \qquad (2.21)$$

where c_g is the group velocity.

Thus, $A = A(\xi,\tau)$ and the function ψ in (2.5) now has the form

$$\begin{aligned}\psi_1(x,\xi,y,t,\tau) &= A(\xi,\tau)\varphi(y)\mathrm{e}^{ik(x-ct)} + A^*(\xi,\tau)\varphi^*(y)\mathrm{e}^{-ik(x-ct)} \\ &= A(\xi,\tau)\varphi(y)\mathrm{e}^{ik(x-ct)} + c.c.\ ,\end{aligned} \qquad (2.22)$$

where the abbreviation c.c. means the complex conjugate.

The stream function in (2.5) can be represented as follows:

$$\psi = \psi(x, y, t, \xi(x,t), \tau(t)). \qquad (2.23)$$

Using the chain rule we can rewrite the derivatives of ψ with respect to t and x in the form

$$\frac{\partial\psi(x,y,t,\xi(x,t),\tau(t))}{\partial t} = \frac{\partial\psi}{\partial t} - \varepsilon\cdot c_g\frac{\partial\psi}{\partial\xi} + \varepsilon^2\frac{\partial\psi}{\partial\tau},$$

$$\frac{\partial\psi(x,y,t,\xi(x,t),\tau(t))}{\partial x} = \frac{\partial\psi}{\partial x} + \varepsilon\frac{\partial\psi}{\partial\xi}.$$

In other words, the differential operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ are replaced by

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon \cdot c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}, \tag{2.24}$$

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \cdot \frac{\partial}{\partial \xi}. \tag{2.25}$$

A perturbed solution ψ is sought in the form

$$\psi = \psi_0(y) + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + ..., \tag{2.26}$$

then

$$\psi_x \Rightarrow \psi_x + \varepsilon\psi_\xi$$

$$\psi_t \Rightarrow \psi_t - c_g \varepsilon\psi_\xi + \varepsilon^2\psi_\tau$$

$$\psi_{xy} \Rightarrow \psi_{xy} + \varepsilon\psi_{\xi y}$$

$$\psi_{xxy} \Rightarrow \psi_{xxy} + 2\varepsilon\psi_{xy\xi} + \varepsilon^2\psi_{y\xi\xi}$$

$$\psi_{xyy} \Rightarrow \psi_{xyy} + \varepsilon\psi_{\xi yy}$$

$$\psi_{xx} \Rightarrow (\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial \xi})(\psi_x + \varepsilon\psi_\xi) = \psi_{xx} + 2\varepsilon\psi_{x\xi} + \varepsilon^2\psi_{\xi\xi}$$

$$\psi_{xxx} \Rightarrow (\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial \xi})(\psi_{xx} + 2\varepsilon\psi_\xi + \varepsilon^2\psi_{\xi\xi}) = \psi_{xxx} + 3\varepsilon\psi_{xx\xi} + 3\varepsilon^2\psi_{x\xi\xi} + \varepsilon^3\psi_{\xi\xi\xi}$$

$$\left(\psi_{xx} + \psi_{yy}\right)_t \Rightarrow (\frac{\partial}{\partial t} - c_g\varepsilon\frac{\partial}{\partial \xi} + \varepsilon^2\frac{\partial}{\partial \tau})(\psi_{xx} + 2\varepsilon\psi_{x\xi} + \varepsilon^2\psi_{\xi\xi} + \psi_{yy})$$

$$= \varepsilon\left(\psi_{1xxt} + \psi_{1yyt}\right) + \varepsilon^2\begin{pmatrix}\psi_{2xxt} - c_g\psi_{1xx\xi} \\ + 2\psi_{1x\xi} \\ + \psi_{2yyt} + c_g\psi_{1yy\xi}\end{pmatrix} + \varepsilon^3\begin{pmatrix}\psi_{3xxt} - c_g\psi_{2xx\xi} + \psi_{1xx\tau} \\ + 2\psi_{2x\xi} - 2c_g\psi_{1x\xi\xi} + \psi_{1\xi\xi} \\ + \psi_{3yyt} - c_g\psi_{2yy\xi} + \psi_{1yy\tau}\end{pmatrix}$$

$$\psi_x^2 + \psi_y^2 \Rightarrow \left(\psi_x + \varepsilon\psi_\xi\right)^2 + \psi_y^2 = \varepsilon^2\psi_{1x}^2 + 2\varepsilon^2(\psi_{2x} + \psi_{1\xi})\psi_{1x} + \psi_{0y}^2 + 2\varepsilon\psi_{0y}\psi_{1y}$$
$$+ \varepsilon^2\left(2\psi_{0y}\psi_{2y} + \psi_{1y}^2\right) + 2\varepsilon^3\left(\psi_{1y}\psi_{2y} + \psi_{0y}\psi_{3y}\right)$$

$$\psi_y\psi_{xy} \Rightarrow \left(\psi_{0y} + \varepsilon\psi_{1y} + \varepsilon^2\psi_{2y} + \varepsilon^3\psi_{3y}\right)\begin{pmatrix}\psi_{0xy} + \varepsilon\psi_{1xy} + \varepsilon^2\psi_{2xy} + \varepsilon^3\psi_{3xy} \\ + \varepsilon\psi_{0\xi y} + \varepsilon^2\psi_{1\xi y} + \varepsilon^3\psi_{2\xi y}\end{pmatrix}$$

$$= \varepsilon\psi_{0y}\psi_{1xy} + \varepsilon^2(\psi_{0y}\psi_{2xy} + \psi_{0y}\psi_{1\xi y} + \psi_{1y}\psi_{1xy}) + \varepsilon^3\begin{pmatrix}\psi_{0y}\psi_{3xy} + \psi_{0y}\psi_{2\xi y} \\ \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi y} + \psi_{2y}\psi_{1xy}\end{pmatrix}$$

Using the following formulas

$$\omega = \sqrt{1 + Ax + Bx^2 + Cx^3}, \qquad \omega(0) = 1$$

$$\omega' = \frac{A + 2Bx + 3Cx^2}{2\sqrt{A + 2Bx + 3Cx^2}}, \qquad \omega'(0) = \frac{A}{2}$$

$$\omega'' = -\frac{A + 2Bx + 3Cx^2}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^3}} + \frac{2B + 6Cx}{2\sqrt{1 + Ax + Bx^2 + Cx^3}} \qquad \omega''(0) = B - \frac{A^2}{4}$$

$$\omega''' = \frac{3(A + 2Bx + 3Cx^2)^3}{8\sqrt{(1 + Ax + Bx^2 + Cx^3)^5}} - \frac{2(A + 2Bx + 3Cx^2)(2B + 6Cx)}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^3}} -$$
$$- \frac{(A + 2Bx + 3Cx^2)(2B + 6Cx)}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^3}} + \frac{6C}{\sqrt{1 + Ax + Bx^2 + Cx^3}}$$

$$\omega'''(0) = \frac{3A^3 - 12AB + 24C}{8}$$

we obtain

$$\sqrt{1 + Ax + Bx^2 + Cx^3} = 1 + \frac{A}{2}x + \frac{\left(4B - A^2\right)}{8}x^2 + \frac{\left(A^3 - 4AB + 8C\right)}{16}x^3$$

$$\sqrt{1 + A\varepsilon + B\varepsilon^2 + C\varepsilon^3} = 1 + \frac{A}{2}\varepsilon + \frac{(4B - A^2)}{8}\varepsilon^2 + \frac{\left(A^3 - 4AB + 8C\right)}{16}\varepsilon^3$$

Thus,

$$\sqrt{\psi_x^2 + \psi_y^2} \Rightarrow$$
$$\sqrt{\psi_{0y}^2 + 2\varepsilon\psi_{0y}\psi_{1y} + \varepsilon^2\left(2\psi_{0y}\psi_{2y} + \psi_{1y}^2 + \psi_{1x}^2\right) + 2\varepsilon^3\left(\psi_{2x}\psi_{1x} + \psi_{1x}\psi_{1\xi} + \psi_{1y}\psi_{2y} + \psi_{0y}\psi_{3y}\right)}$$
$$= \psi_{0y}\sqrt{1 + 2\varepsilon\frac{\psi_{1y}}{\psi_{0y}} + \frac{\varepsilon^2\left(2\psi_{0y}\psi_{2y} + \psi_{1y}^2 + \psi_{1x}^2\right)}{\psi_{0y}^2} + \frac{2\varepsilon^3\left(\psi_{2x}\psi_{1x} + \psi_{1x}\psi_{1\xi} + \psi_{1y}\psi_{2y} + \psi_{0y}\psi_{3x}\right)}{\psi_{0y}^2}}$$
$$= \psi_{0y}\left(\begin{array}{l} 1 + \dfrac{2\psi_{1y}}{\psi_{0y}2}\varepsilon + \left(\dfrac{4\left(2\psi_{0y}\psi_{2y} + \psi_{1y}^2 + \psi_{1x}^2\right)}{\psi_{0y}^2} - \dfrac{4\psi_{1y}^2}{\psi_{0y}^2}\right)\dfrac{\varepsilon^2}{8} \\ + \left(\dfrac{8\psi_{1y}^3}{\psi_{0y}^3} - \dfrac{8\psi_{1y}\left(2\psi_{0y}\psi_{2y} + \psi_{1y}^2 + \psi_{1x}^2\right)}{\psi_{0y}\psi_{0y}^2} + \dfrac{16\left(\psi_{2x}\psi_{1x} + \psi_{1x}\psi_{1\xi} + \psi_{1y}\psi_{2y} + \psi_{0y}\right)}{\psi_{0y}^2}\right)\dfrac{\varepsilon^3}{16} \end{array}\right)$$
$$= \psi_{0y} + \varepsilon\psi_{1y} + \varepsilon^2\left(\psi_{2y} + \frac{\psi_{1x}^2}{2\psi_{0y}}\right) + \varepsilon^3\left(-\frac{\psi_{1y}\psi_{1x}^2}{2\psi_{0y}^2} + \frac{\psi_{1x}\psi_{2x}}{\psi_{0y}} + \frac{\psi_{1x}\psi_{1\xi}}{\psi_{0y}} + \psi_{3y}\right).$$

Similarly:

$$\omega = \frac{1}{\sqrt{1 + Ax + Bx^2 + Cx^3}}, \qquad \omega(0) = 1$$

$$\omega' = -\frac{A + 2Bx + 3Cx^2}{2\sqrt{(A + 2Bx + 3Cx^2)^3}}, \qquad \omega'(0) = -\frac{A}{2}$$

$$\omega'' = \frac{3(A + 2Bx + 3Cx^2)^2}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^5}} - \frac{2B + 6Cx}{2\sqrt{(1 + Ax + Bx^2 + Cx^3)^3}},$$

$$\omega''(0) = \frac{3A^2 - 4B}{4}$$

$$\omega''' = -\frac{15(A + 2Bx + 3Cx^2)^3}{8\sqrt{(1 + Ax + Bx^2 + Cx^3)^7}} + \frac{3 \cdot 2(A + 2Bx + 3Cx^2)(2B + 6Cx)}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^5}} + \frac{3(A + 2Bx + 3Cx^2)(2B + 6Cx)}{4\sqrt{(1 + Ax + Bx^2 + Cx^3)^5}} - \frac{6C}{2\sqrt{(1 + Ax + Bx^2 + Cx^3)^3}},$$

$$\omega'''(0) = -\frac{15}{8}A^3 + \frac{9}{2}AB - 3C$$

$$\frac{1}{\sqrt{1 + Ax + Bx^2 + Cx^3}} = 1 - \frac{A}{2}x + \frac{3A^2 - 4B}{8}x^2 + \frac{\left(-5A^3 + 12AB - 8C\right)}{16}x^3.$$

Hence,

$$\frac{1}{\sqrt{\psi_x^2 + \psi_y^2}} \Rightarrow \frac{1}{\psi_{0y}}\left(\begin{array}{l} 1 - \dfrac{\psi_{1y}}{\psi_{0y}}\varepsilon + \dfrac{2\psi_{1y}^2 - 2\psi_{0y}\psi_{2y} - \psi_{1x}^2}{2\psi_{0y}^2}\varepsilon^2 \\ + \dfrac{\varepsilon^3}{2\psi_{0y}^3}\left(\begin{array}{l} -2\psi_{1y}^3 + 4\psi_{0y}\psi_{1y}\psi_{2y} + 3\psi_{1y}\psi_{1x}^2 - 2\psi_{0y}\psi_{1x}\psi_{2x} \\ -2\psi_{0y}\psi_{1x}\psi_{1\xi} - 2\psi_{0y}^2\psi_{3y} \end{array}\right) \end{array}\right)$$

$$\begin{aligned}
&\psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y \Rightarrow \psi_y(\psi_{xx} + \psi_{yy})_x - \psi_x(\psi_{xx} + \psi_{yy})_y \\
&= \varepsilon(\psi_{0y}\psi_{1xxx} + \psi_{0y}\psi_{1yyx} - \psi_{1x}\psi_{0yyy}) \\
&+ \varepsilon^2\left(\begin{array}{l} \psi_{0y}\psi_{2xxx} + 3\psi_{0y}\psi_{1xx\xi} + \psi_{1y}\psi_{1xxx} + \psi_{1y}\psi_{1yyx} + \psi_{0y}\psi_{2yyx} \\ + \psi_{0y}\psi_{1\xi yy} - \psi_{1x}\psi_{1xxy} - \psi_{1x}\psi_{1yyy} - \psi_{2x}\psi_{0yyy} - \psi_{1\xi}\psi_{0yyy} \end{array}\right) \\
&+ \varepsilon^3\left(\begin{array}{l} \psi_{0y}\psi_{3xxx} + 3\psi_{0y}\psi_{2xx\xi} + 3\psi_{0y}\psi_{1x\xi\xi} + \psi_{1y}\psi_{2xxx} + 3\psi_{1y}\psi_{1xx\xi} + \psi_{2y}\psi_{1xxx} + \psi_{2y}\psi_{1yyx} \\ + \psi_{1y}\psi_{2yyx} + \psi_{0y}\psi_{3yyx} + \psi_{1y}\psi_{1\xi yy} + \psi_{0y}\psi_{2\xi yy} - \psi_{2x}\psi_{1xxy} - \psi_{1\xi}\psi_{1xxy} - \psi_{1x}\psi_{2xxy} \\ - 2\psi_{1x}\psi_{1xy\xi} - \psi_{1x}\psi_{2yyy} - \psi_{2x}\psi_{1yyy} - \psi_{3x}\psi_{0yyy} - \psi_{1\xi}\psi_{1yyy} - \psi_{2\xi}\psi_{0yyy} \end{array}\right)
\end{aligned}$$

$$
\begin{aligned}
&\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx} \Rightarrow \psi_{0y}^2\psi_{0yy} + \varepsilon\left(2\psi_{0y}\psi_{0yy}\psi_{1y} + \psi_{1yy}\psi_{0y}^2\right) \\
&+\varepsilon^2\begin{pmatrix} \psi_{1y}^2\psi_{0yy} + 2\psi_{0y}\psi_{2y}\psi_{0yy} + 2\psi_{0y}\psi_{1y}\psi_{1yy} \\ +\psi_{0y}^2\psi_{2yy} + 2\psi_{1x}\psi_{0yx}\psi_{1xy} \end{pmatrix} \\
&+\varepsilon^3\begin{pmatrix} 2\psi_{0y}\psi_{3y}\psi_{0yy} + 2\psi_{1y}\psi_{2y}\psi_{0yy} + \psi_{1y}^2\psi_{1yy} + 2\psi_{0y}\psi_{2y}\psi_{1yy} + 2\psi_{0y}\psi_{1y}\psi_{2yy} \\ +\psi_{0y}^2\psi_{3yy} + 2\psi_{1x}\psi_{0y}\psi_{2xy} + 2\psi_{1x}\psi_{0y}\psi_{1\xi y} + 2\psi_{2x}\psi_{0y}\psi_{1xy} \\ +2\psi_{1\xi}\psi_{0y}\psi_{1xy} + 2\psi_{1y}\psi_{1x}\psi_{1xy} + \psi_{1x}^2\psi_{1xx} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\psi = \psi_{xx} + \psi_{yy} \Rightarrow \psi_{xx} + 2\varepsilon\psi_{x\xi} + \varepsilon^2\psi_{\xi\xi} + \psi_{yy} \\
&= \psi_{0xx} + \varepsilon\psi_{1xx} + \varepsilon^2\psi_{2xx} + \varepsilon^3\psi_{3xx} + 2\varepsilon\psi_{0x\xi} + 2\varepsilon^2\psi_{1x\xi} + 2\varepsilon^3\psi_{2x\xi} \\
&+ \varepsilon^2\psi_{0\xi\xi} + \varepsilon^3\psi_{1\xi\xi} + \psi_{0yy} + \varepsilon\psi_{1yy} + \varepsilon^2\psi_{2yy} + \varepsilon^3\psi_{3yy}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) \\
&= \frac{c_{\mathrm{f}}}{2h}\left(\begin{array}{l}
\psi_{0y}\psi_{0yy} + \varepsilon\left(\psi_{0yy}\psi_{1y} + \psi_{1yy}\psi_{0y}\right) + \varepsilon^2\begin{pmatrix} \psi_{2y}\psi_{0yy} + \psi_{1y}\psi_{1yy} + \psi_{0y}\psi_{2yy} \\ + 2\psi_{1x}\psi_{1xy} - \dfrac{\psi_{0yy}\psi_{1x}^2}{2\psi_{0y}} - \psi_{0y}\psi_{0yy} \end{pmatrix} \\
+\varepsilon^3\begin{pmatrix} \dfrac{\psi_{0yy}\psi_{1y}\psi_{1x}^2}{2\psi_{0y}^2} - \dfrac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}} - \dfrac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}} - \psi_{0yy}\psi_{3y} + \psi_{1yy}\psi_{2y} \\ -\dfrac{\psi_{1yy}\psi_{1x}^2}{2\psi_{0y}} + \psi_{1y}\psi_{2yy} + 2\psi_{3y}\psi_{0yy} + \psi_{0y}\psi_{3yy} + 2\psi_{1x}\psi_{2xy} \\ + 2\psi_{1x}\psi_{1\xi\ y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy} + \dfrac{\psi_{1x}^2\psi_{1xx}}{\psi_{0y}} - \psi_{0yy}\psi_{1y} - \psi_{1yy}\psi_{0y} \end{pmatrix}
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} \\
&= \frac{c_{\mathrm{f}}}{2h}\left(\begin{array}{l}
\psi_{0y}\psi_{0yy} + \varepsilon\begin{pmatrix} \psi_{1xx}\psi_{0y} + \psi_{0yy}\psi_{1y} \\ + \psi_{0y}\psi_{1yy} \end{pmatrix} + \varepsilon^2\begin{pmatrix} \psi_{1xx}\psi_{1y} + \psi_{2xx}\psi_{0y} + 2\psi_{2x\xi}\psi_{0y} \\ + \psi_{0yy}\psi_{2y} + \dfrac{\psi_{0yy}\psi_{1x}^2}{2\psi_{0y}} + \psi_{1yy}\psi_{1y} \\ + \psi_{2yy}\psi_{0y} - \psi_{0y}\psi_{0yy} \end{pmatrix} \\
+\varepsilon^3\begin{pmatrix} \psi_{1xx}\psi_{2y} + \dfrac{\psi_{1xx}\psi_{1x}^2}{2\psi_{0y}} + \psi_{2xx}\psi_{1y} + \psi_{3xx}\psi_{0y} + 2\psi_{1x\xi}\psi_{1y} + 2\psi_{0y}\psi_{2x\xi} \\ + \psi_{1\xi\xi}\psi_{0y} - \dfrac{\psi_{0yy}\psi_{1y}\psi_{1x}^2}{2\psi_{0y}^2} + \dfrac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}} + \dfrac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}} + \psi_{0yy}\psi_{3y} \\ + \psi_{1yy}\psi_{2y} + \psi_{1yy}\dfrac{\psi_{1x}^2}{2\psi_{0y}} + \psi_{2yy}\psi_{1y} + \psi_{3yy}\psi_{0y} - \psi_{1xx}\psi_{0y} - \psi_{0yy}\psi_{1y} - \psi_{0y}\psi_{1yy} \end{pmatrix}
\end{array}\right)
\end{aligned}
$$

Substituting all expressions into (2.2) we obtain the following equation:

$$
\begin{aligned}
&\varepsilon\left(\psi_{1xxt}+\psi_{1yyt}\right)+\varepsilon^{2}\left(\begin{array}{l}\psi_{2xxt}-c_{g}\psi_{1xx\xi}\\ +2\psi_{1x\xi t}\\ +\psi_{2yyt}+c_{g}\psi_{1yy\xi}\end{array}\right)+\varepsilon^{3}\left(\begin{array}{l}\psi_{3xxt}-c_{g}\psi_{2xx\xi}+\psi_{1xx\tau}\\ +2\psi_{2x\xi t}-2c_{g}\psi_{1x\xi\xi}+\psi_{1\xi\xi t}\\ +\psi_{3yyt}-c_{g}\psi_{2yy\xi}+\psi_{1yy\tau}\end{array}\right)\\
&+\varepsilon(\psi_{0y}\psi_{1xxx}+\psi_{0y}\psi_{1yyx}-\psi_{1x}\psi_{0yyy})\\
&+\varepsilon^{2}\left(\begin{array}{l}\psi_{0y}\psi_{2xxx}+3\psi_{0y}\psi_{1xx\xi}+\psi_{1y}\psi_{1xxx}+\psi_{1y}\psi_{1yyx}+\psi_{0y}\psi_{2yyx}\\ +\psi_{0y}\psi_{1\xi yy}-\psi_{1x}\psi_{1xxy}-\psi_{1x}\psi_{1yyy}-\psi_{2x}\psi_{0yyy}-\psi_{1\xi}\psi_{0yyy}\end{array}\right)\\
&+\varepsilon^{3}\left(\begin{array}{l}\psi_{0y}\psi_{3xxx}+3\psi_{0y}\psi_{2xx\xi}+3\psi_{0y}\psi_{1x\xi\xi}+\psi_{1y}\psi_{2xxx}+3\psi_{1y}\psi_{1xx\xi}+\psi_{2y}\psi_{1xxx}\\ +\psi_{2y}\psi_{1yyx}+\psi_{1y}\psi_{2yyx}+\psi_{0y}\psi_{3yyx}+\psi_{1y}\psi_{1\xi yy}+\psi_{0y}\psi_{2\xi yy}-\psi_{2x}\psi_{1xxy}\\ -\psi_{1\xi}\psi_{1xxy}-\psi_{1x}\psi_{2xxy}-2\psi_{1x}\psi_{1xy\xi}-\psi_{1x}\psi_{2yyy}-\psi_{2x}\psi_{1yyy}-\psi_{3x}\psi_{0yyy}\\ -\psi_{1\xi}\psi_{1yyy}-\psi_{2\xi}\psi_{0yyy}\end{array}\right)\\
&+\frac{c_{\mathrm{f}}}{2h}\left(\begin{array}{l}\psi_{0y}\psi_{0yy}+\varepsilon\left(\begin{array}{l}\psi_{0yy}\psi_{1y}\\ +\psi_{1yy}\psi_{0y}\end{array}\right)+\varepsilon^{2}\left(\begin{array}{l}\psi_{2y}\psi_{0yy}+\psi_{1y}\psi_{1yy}+\psi_{0y}\psi_{2yy}\\ +2\psi_{1x}\psi_{1xy}-\dfrac{\psi_{0yy}\psi_{1x}^{2}}{2\psi_{0y}}-\psi_{0y}\psi_{0yy}\end{array}\right)\\
+\varepsilon^{3}\left(\begin{array}{l}\dfrac{\psi_{0yy}\psi_{1y}\psi_{1x}^{2}}{2\psi_{0y}^{2}}-\dfrac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}}-\dfrac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}}-\psi_{0yy}\psi_{3y}+\psi_{1yy}\psi_{2y}\\ -\dfrac{\psi_{1yy}\psi_{1x}^{2}}{2\psi_{0y}}+\psi_{1y}\psi_{2yy}+2\psi_{3y}\psi_{0yy}+2\psi_{0y}\psi_{3yy}+2\psi_{1x}\psi_{2xy}\\ +2\psi_{1x}\psi_{1\xi y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}+\dfrac{\psi_{1x}^{2}\psi_{1xx}}{\psi_{0y}}-\psi_{0yy}\psi_{1y}-\psi_{1yy}\psi_{0y}\end{array}\right)\end{array}\right)\\
&+\frac{c_{\mathrm{f}}}{2h}\left(\begin{array}{l}\psi_{0y}\psi_{0yy}+\varepsilon\left(\begin{array}{l}\psi_{1xx}\psi_{0y}\\ +\psi_{0yy}\psi_{1y}+\psi_{0y}\psi_{1yy}\end{array}\right)+\varepsilon^{2}\left(\begin{array}{l}\psi_{1xx}\psi_{1y}+\psi_{2xx}\psi_{0y}\\ +2\psi_{2x\xi}\psi_{0y}+\psi_{0yy}\psi_{2y}\\ +\dfrac{\psi_{0yy}\psi_{1x}^{2}}{2\psi_{0y}}+\psi_{1yy}\psi_{1y}\\ +\psi_{2yy}\psi_{0y}-\psi_{0y}\psi_{0yy}\end{array}\right)\\
+\varepsilon^{3}\left(\begin{array}{l}\psi_{1xx}\psi_{2y}+\dfrac{\psi_{1xx}\psi_{1x}^{2}}{2\psi_{0y}}+\psi_{2xx}\psi_{1y}+\psi_{3xx}\psi_{0y}+2\psi_{1x\xi}\psi_{1y}+2\psi_{0y}\psi_{2x\xi}\\ +\psi_{1\xi\xi}\psi_{0y}-\dfrac{\psi_{0yy}\psi_{1y}\psi_{1x}^{2}}{2\psi_{0y}^{2}}+\dfrac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}}+\dfrac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}}\\ +\psi_{0yy}\psi_{3y}+\psi_{1yy}\psi_{2y}+\psi_{1yy}\dfrac{\psi_{1x}^{2}}{2\psi_{0y}}+\psi_{2yy}\psi_{1y}+\psi_{3yy}\psi_{0y}\\ -\psi_{1xx}\psi_{0y}-\psi_{0yy}\psi_{1y}-\psi_{0y}\psi_{1yy}\end{array}\right)\end{array}\right)\\
&+\frac{2}{R}\left(\begin{array}{l}\varepsilon\psi_{0y}\psi_{1xy}+\varepsilon^{2}(\psi_{0y}\psi_{2xy}+\psi_{0y}\psi_{1y\xi}+\psi_{1y}\psi_{1xy})\\ +\varepsilon^{3}(\psi_{0y}\psi_{3xy}+\psi_{0y}\psi_{2y\xi}+\psi_{1y}\psi_{2xy}+\psi_{1y}\psi_{1y\xi}+\psi_{2y}\psi_{1xy})\end{array}\right)=0. \qquad (2.27)
\end{aligned}
$$

Collecting the terms of orders ε, ε^2, ε^3 we obtain the following three equations:

$$\begin{aligned}&\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}\psi_{1xxx}+\psi_{0y}\psi_{1yyx}-\psi_{1x}\psi_{0yyy}+\\&+\frac{2}{R}\psi_{0y}\psi_{1xy}+\frac{c_{\mathrm{f}}}{2h}\left(\psi_{1xx}\psi_{0y}+2\psi_{0yy}\psi_{1y}+2\psi_{0y}\psi_{1yy}\right)=0\ .\end{aligned} \tag{2.28}$$

$$\begin{aligned}&\psi_{2xxt}-c_g\psi_{1xx\xi}+2\psi_{1x\xi t}+\psi_{2yyt}-c_g\psi_{1yy\xi}+\psi_{0y}\psi_{2xxx}+3\psi_{0y}\psi_{1xx\xi}+\psi_{1y}\psi_{1xxx}\\&+\psi_{1y}\psi_{1yyx}+\psi_{0y}\psi_{2yyx}+\psi_{0y}\psi_{1\xi yy}-\psi_{1x}\psi_{1xxy}-\psi_{1x}\psi_{1yyy}-\psi_{2x}\psi_{0yyy}-\psi_{1\xi}\psi_{0yyy}\\&+\frac{2}{R}\left(\psi_{0y}\psi_{2xy}+\psi_{0y}\psi_{1\xi y}+\psi_{1y}\psi_{1xy}\right)\\&+\frac{c_{\mathrm{f}}}{2h}\left(\begin{aligned}&\psi_{1xx}\psi_{1y}+\psi_{2xx}\psi_{0y}+2\psi_{1x\xi}\psi_{0y}+\psi_{0yy}\psi_{2y}+\frac{\psi_{0yy}\psi_{1x}^2}{2\psi_{0y}}+\psi_{1yy}\psi_{1y}\\&+\psi_{2yy}\psi_{0y}-\psi_{0y}\psi_{0yy}+\psi_{2y}\psi_{0yy}+\psi_{1y}\psi_{1yy}\\&+\psi_{0y}\psi_{2yy}+2\psi_{1x}\psi_{1xy}-\frac{\psi_{0yy}\psi_{1x}^2}{2\psi_{0y}}-\psi_{0y}\psi_{0yy}\end{aligned}\right)=0.\end{aligned} \tag{2.29}$$

$$\begin{aligned}&\psi_{3xxt}-c_g\psi_{2xx\xi}+\psi_{1xx\tau}+2\psi_{2x\xi t}-2c_g\psi_{1x\xi\xi}+\psi_{1\xi\xi t}+\psi_{3yyt}-c_g\psi_{2yy\xi}+\psi_{1yy\tau}\\&\psi_{0y}\psi_{3xxx}+3\psi_{0y}\psi_{2xx\xi}+3\psi_{0y}\psi_{1x\xi\xi}+\psi_{1y}\psi_{2xxx}+3\psi_{1y}\psi_{1xx\xi}+\psi_{2y}\psi_{1xxx}\\&+\psi_{2y}\psi_{1yyx}+\psi_{1y}\psi_{2yyx}+\psi_{0y}\psi_{3yyx}+\psi_{1y}\psi_{1\xi yy}+\psi_{0y}\psi_{2\xi yy}-\psi_{2x}\psi_{1xxy}-\psi_{1\xi}\psi_{1xxy}\\&-\psi_{1x}\psi_{2xxy}-2\psi_{1x}\psi_{1xy\xi}-\psi_{1x}\psi_{2yyy}-\psi_{2x}\psi_{1yyy}-\psi_{3x}\psi_{0yyy}-\psi_{1\xi}\psi_{1yyy}-\psi_{2\xi}\psi_{0yyy}\\&+\frac{2}{R}\left(\psi_{0y}\psi_{3xy}+\psi_{0y}\psi_{2\xi y}+\psi_{1y}\psi_{2xy}+\psi_{1y}\psi_{1\xi y}+\psi_{2y}\psi_{1xy}\right)\\&+\frac{c_{\mathrm{f}}}{2h}\left(\begin{aligned}&\psi_{1xx}\psi_{2y}+\frac{\psi_{1xx}\psi_{1x}^2}{2\psi_{0y}}+\psi_{2xx}\psi_{1y}+\psi_{3xx}\psi_{0y}+2\psi_{1x\xi}\psi_{1y}+2\psi_{0y}\psi_{2x\xi}\\&-\frac{\psi_{0yy}\psi_{1y}\psi_{1x}^2}{2\psi_{0y}^2}+\frac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}}+\frac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}}+\psi_{0yy}\psi_{3y}+\psi_{1yy}\psi_{2y}\\&+\psi_{yy}\,\frac{\psi_{1x}^2}{2\psi_{0y}}+\psi_{2yy}\psi_{1y}+\psi_{3yy}\psi_{0y}-\psi_{1xx}\psi_{0y}-\psi_{0yy}\psi_{1y}-\psi_{0y}\psi_{1yy}\\&+\frac{\psi_{0yy}\psi_{1y}\psi_{1x}^2}{2\psi_{0y}^2}-\frac{\psi_{0yy}\psi_{1x}\psi_{2x}}{\psi_{0y}}-\frac{\psi_{0yy}\psi_{1x}\psi_{1\xi}}{\psi_{0y}}-\psi_{0yy}\psi_{3y}+\psi_{1yy}\psi_{2y}\\&-\frac{\psi_{1yy}\psi_{1x}^2}{2\psi_{0y}}+\psi_{1y}\psi_{2yy}+2\psi_{3y}\psi_{0yy}+\psi_{0y}\psi_{3yy}+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi y}\\&+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}+\frac{\psi_{1x}^2\psi_{1xx}}{\psi_{0y}}-\psi_{0yy}\psi_{1y}-\psi_{1yy}\psi_{0y}+\psi_{1\xi\xi}\psi_{0y}\end{aligned}\right)=0\end{aligned} \tag{2.30}$$

Let

$$
\begin{aligned}
L\varphi \equiv\ & \varphi_{xxt} + \varphi_{yyt} + \varphi_{0y}\varphi_{xxx} + \varphi_{0y}\varphi_{yyx} - \varphi_{0yyy}\varphi_x \\
& + \frac{2}{R}\varphi_{0y}\varphi_{1xy} + \frac{c_{\mathrm{f}}}{2h}\left(\varphi_{0y}\varphi_{xx} + 2\varphi_{0yy}\varphi_y + 2\varphi_{0y}\varphi_{yy}\right),
\end{aligned}
\tag{2.31}
$$

then equation (2.28) can be rewritten as follows:

$$
L\psi_1 = 0 \ . \tag{2.32}
$$

Using the notation $U = \psi_{0y}$, we rewrite (2.28) in the form

$$
\begin{aligned}
& \psi_{1xxt} + \psi_{1yyt} + U\psi_{1xxx} + U\psi_{1xyy} - U_{yy}\psi_{1x} \\
& + \frac{2}{R}U\psi_{1xy} + \frac{c_{\mathrm{f}}}{2h}\left(U\psi_{1xx} + 2U_y\psi_{1y} + 2U\psi_{1yy}\right) = 0 \ .
\end{aligned}
\tag{2.33}
$$

Equation (2.29) is rewritten in the form

$$
\begin{aligned}
L\psi_2 =\ & c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi\ t} - 3U\psi_{1xx\xi} - \psi_{1y}\psi_{1xxx} \\
& - \psi_{1y}\psi_{1yyx} - U\psi_{1\xi\ yy} + \psi_{1x}\psi_{1xxy} + \psi_{1x}\psi_{1yyy} + U_{yy}\psi_{1\xi} \\
& - \frac{c_{\mathrm{f}}}{2h}\left(\psi_{1xx}\psi_{1y} + 2U\psi_{1x\xi} + 2\psi_{1yy}\psi_{1y} - 2UU_y + 2\psi_{1x}\psi_{1xy}\right) \\
& - \frac{2}{R}\left(U\psi_{1\xi\ y} + \psi_{1y}\psi_{1xy}\right) .
\end{aligned}
\tag{2.34}
$$

Note that the operator L on the left-hand side of (2.34) is the same as in (2.32) and it will be the same for all orders of ε .

In terms of the operator L equation (2.30) can be rewritten as follows

$$
\begin{aligned}
L\psi_3 =\ & c_g(\psi_{2xx\xi} + \psi_{2yy\xi}) - \psi_{1xx\tau} - 2\psi_{2x\xi\ t} + 2c_g\psi_{1x\xi\xi} - \psi_{1\xi\xi\ t} - \psi_{1yy\tau} \\
& - 3U\psi_{2xx\xi} - 3U\psi_{1x\xi\xi} - \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \psi_{2y}\psi_{1xxx} - \psi_{2y}\psi_{1yyx} \\
& - \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi\ yy} - U\psi_{2\xi\ yy} + \psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} \\
& + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} + \psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}U_{yy} \\
& - \frac{c_{\mathrm{f}}}{2h}\left(
\begin{aligned}
& \psi_{1xx}\psi_{2y} + \frac{3\psi_{1xx}\psi_{1x}^2}{2U} + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2U\psi_{2x\xi} \\
& + U\psi_{1\xi\xi} + 2\psi_{1yy}\psi_{2y} + 2\psi_{2yy}\psi_{1y} - U\psi_{1xx} - 2U_y\psi_{1y} \\
& - 2U\psi_{1yy} + 2\psi_{1x}\psi_{2xy} + 2\psi_{1x}\psi_{1\xi\ y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}
\end{aligned}
\right) \\
& - \frac{2}{R}\left(U\psi_{2\xi\ y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi\ y} + \psi_{1xy}\psi_{2y}\right).
\end{aligned}
\tag{2.35}
$$

First, we solve the linear stability problem. In equation (2.33) the solution will be sought in the form $\psi_1 = \varphi_1(y)\mathrm{e}^{ik(x-ct)}$.

Substituting derivatives into the equation (2.33) we obtain:

$$\varphi_1''\left(Uik - ikc + \frac{c_\mathrm{f}}{h}U\right) + \varphi_1'\left(\frac{2}{R}Uik + \frac{c_\mathrm{f}}{h}U_y\right) + \varphi_1\left(ik^3c - Uik^3 - ik_1U_{yy} - \frac{c_\mathrm{f}}{2h}Uk^2\right) = 0$$

or

$$\varphi_1''\left(U - c - \frac{iSU}{k}\right) + \varphi_1'\left(\frac{2}{R}U - \frac{iSU_y}{k}\right) + \varphi_1\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2}\right) = 0 \quad (2.36)$$

where $S = \dfrac{c_\mathrm{f}\delta_*}{h}$ – the bed-friction number;

δ_* – the width of the mixing layer.

The boundary conditions are

$$\varphi_1(\pm\infty) = 0. \quad (2.37)$$

Numerical solution of (2.36), (2.37) is obtained in Section 2.2. We can find the critical values of the S_c, k_c and c_c (stability parameter, wave number and wave speed, respectively).

Assume now ψ_1 in the form (2.22). Next, we consider the solution of (2.34). Derivatives of the right side of the equation are:

$$\psi_{1x} = A\varphi_1 ike^{ik(x-ct)} - ikA^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1\xi} = A_\xi\varphi_1\mathrm{e}^{ik(x-ct)} + A_\xi^{\,*}\varphi_1^{\,*}\mathrm{e}^{-ik(x-ct)}$$

$$\psi_{1\xi\,y} = A_\xi\varphi_{1y}\mathrm{e}^{ik(x-ct)} + A_\xi^{\,*}\varphi_{1y}^{\,*}\mathrm{e}^{-ik(x-ct)}$$

$$\psi_{1xx} = -k^2A\varphi_1 e^{ik(x-ct)} - k^2A^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xx\xi} = -k^2A_\xi\varphi_1 e^{ik(x-ct)} - k^2A_\xi^{\,*}\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xxt} = ik^3cA\varphi_1 e^{ik(x-ct)} - ik^3cA^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xxy} = -k^2A\varphi_{1y} e^{ik(x-ct)} - k^2A^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1xyy} = A\varphi_{1yy} ike^{ik(x-ct)} - ikA^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1x\xi\,t} = k^2cA_\xi\varphi_1 e^{ik(x-ct)} + k^2cA_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xxx} = -ik^3A\varphi_1 e^{ik(x-ct)} + ik^3A^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1y} = A\varphi_{1y} e^{ik(x-ct)} + A^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1yy} = A\varphi_{1yy} e^{ik(x-ct)} + A^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1yy\xi} = A_\xi\varphi_{1yy} e^{ik(x-ct)} + A_\xi^{\,*}\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1xyy} = A\varphi_{1yy} ike^{ik(x-ct)} - ikA^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1yyy} = A\varphi_{1yyy} e^{ik(x-ct)} + A^*\varphi_{1yyy}^* e^{-ik(x-ct)}$$

$$\psi_{1xy} = ikA\varphi_{1y} e^{ik(x-ct)} - ikA^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1x\xi} = ikA_\xi\varphi_1 e^{ik(x-ct)} - ikA_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

We substitute the derivatives in the right-hand side of the equation (2.34) and simplify:

$$
\frac{c_\mathrm{f}}{2h}2UU_y + AA^*\begin{pmatrix} ik^3\varphi_{1y}\varphi_1^* + ik^3\varphi_{1y}^*\varphi_1 + ik\varphi_{1y}\varphi_{1yy}^* - ik\varphi^*{}_{1y}\varphi_{1yy} - ik^3\varphi_1\varphi^*{}_{1y} \\ -ik^3\varphi_1^*\varphi_{1y} + ik\varphi_1\varphi_{1yyy}^* - ik\varphi^*{}_1\varphi_{1yyy} + \frac{c_\mathrm{f}}{2h}k^2\varphi_1\varphi_{1y}^* + \frac{c_\mathrm{f}}{2h}k^2\varphi_1^*\varphi_{1y} \\ -\frac{c_\mathrm{f}}{2h}2\varphi_{1y}\varphi_{1yy}^* - \frac{c_\mathrm{f}}{2h}2\varphi_{1y}^*\varphi_{1yy} - \frac{c_\mathrm{f}}{2h}2k^2\varphi_{1y}\varphi_1^* - \frac{c_\mathrm{f}}{2h}2k^2\varphi_{1y}^*\varphi_1 \end{pmatrix}
$$

$$
+ A_\xi \mathrm{e}^{ik(x-ct)}\begin{pmatrix} -c_g k^2\varphi_1 + c_g\varphi_{1yy} - 2k^2c\varphi_1 \\ +3Uk^2\varphi_1 - U\varphi_{1yy} + U_{yy}\varphi_1 \\ -\frac{c_\mathrm{f}}{2h}2Uik\varphi_1 - \frac{2}{R}U\varphi_{1y} \end{pmatrix} + A_\xi{}^*\mathrm{e}^{-ik(x-ct)}\begin{pmatrix} -c_g k^2\varphi_1^* + c_g\varphi_{1yy}^* - 2k^2c\varphi_1^* \\ +3Uk^2\varphi_1^* - U\varphi_{1yy}^* + U_{yy}\varphi_1^* \\ +\frac{c_\mathrm{f}}{2h}2Uik\varphi_1^* - \frac{2}{R}U\varphi_{1y}^* \end{pmatrix}
$$

$$
+ A^2\mathrm{e}^{2ik(x-ct)}\begin{pmatrix} ik^3\varphi_{1y}\varphi_1 - ik\varphi_{1y}\varphi_{1yy} \\ +ik\varphi_1\varphi_{1yyy} - ik^3\varphi_1\varphi_{1y} \\ +\frac{c_\mathrm{f}}{2h}k^2\varphi_1\varphi_{1y} - \frac{c_\mathrm{f}}{2h}2\varphi_{1y}\varphi_{1yy} \\ +\frac{c_\mathrm{f}}{2h}2k^2\varphi_{1y}\varphi_1 - \frac{2}{R}ik\varphi_{1y}^2 \end{pmatrix} + A^{*2}\mathrm{e}^{-2ik(x-ct)}\begin{pmatrix} -ik^3\varphi_{1y}^*\varphi_1^* - ik\varphi_{1y}^*\varphi_{1yy}^* \\ -ik\varphi_1^*\varphi_{1yyy}^* + ik^3\varphi_{1y}^*\varphi_1^* \\ +\frac{c_\mathrm{f}}{2h}k^2\varphi_1^*\varphi_{1y}^* - \frac{c_\mathrm{f}}{2h}2\varphi_{1y}^*\varphi_{1yy}^* \\ +\frac{c_\mathrm{f}}{2h}2k^2\varphi_{1y}^*\varphi_1^* - \frac{2}{R}ik\varphi_{1y}^{*2} \end{pmatrix}.
$$

Terms proportional to AA^* have the form:

$$
ik\begin{pmatrix} \varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} \\ +\varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy} \end{pmatrix} - \frac{c_\mathrm{f}}{2h}\begin{pmatrix} k^2\varphi_1\varphi_{1y}^* + k^2\varphi_1^*\varphi_{1y} \\ +2\varphi_{1y}\varphi_{1yy}^* + 2\varphi_{1y}^*\varphi_{1yy} \end{pmatrix}. \tag{2.38}
$$

Similarly, terms proportional to $A_\xi \cdot \mathrm{e}^{ik(x-ct)}$ are as follows:

$$
\left(c_g - U\right)\varphi_{1yy} - \frac{2}{R}U\varphi_{1y} + \left(-c_g k^2 - 2k^2c + 3Uk^2 + U_{yy} - \frac{c_\mathrm{f}}{2h}2Uik\right)\varphi_1. \tag{2.39}
$$

Finally, terms proportional to $A^2 \cdot \mathrm{e}^{2ik(x-ct)}$ have the form**:**

$$
ik\left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) - \frac{c_\mathrm{f}}{2h}\left(-3k^2\varphi_1\varphi_{1y} + 2\varphi_{1y}\varphi_{1yy}\right) - \frac{2}{R}ik\varphi_{1y}^2. \tag{2.40}
$$

The following three groups of terms will emerge:

a) the terms that are independent on time;

b) the terms proportional to the first harmonic $\mathrm{e}^{ik(x-ct)}$ (here and in sequel we drop the subscripts and use the notation $k=k_{\mathrm{c}}$ and $c=c_{\mathrm{c}}$ for convenience);

c) the terms proportional to the second harmonic $\mathrm{e}^{2ik(x-ct)}$.

Thus, the function ψ_2 should also contain the same three groups of terms. More precisely, we seek the solution to (2.34) in the form

$$\begin{aligned}\psi_2 &= AA^*\varphi_2^{(0)}(y)+A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)}+A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)}\\ &+AA^*\varphi_2^{(0)*}(y)+A^*{}_\xi\varphi_2^{(1)*}(y)\mathrm{e}^{-ik(x-ct)}+A^{2*}\varphi_2^{(2)*}(y)\mathrm{e}^{-2ik(x-ct)}\\ &=AA^*\varphi_2^{(0)}(y)+A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)}+A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)}+c.c.,\end{aligned} \tag{2.41}$$

where $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y;

A^* denotes the complex conjugate of A;

the superscript reflects the index of the harmonic component;

the subscript represents the order of approximation.

Derivatives on the left side of the equation (2.34) are:

$$\psi_{2x}=A_\xi\varphi_2^{(1)}ike^{ik(x-ct)}+A^2\varphi_2^{(2)}2ike^{2ik(x-ct)}-A^*{}_\xi\varphi_2^{(1)}ike^{-ik(x-ct)}-A^{2*}\varphi_2^{(2)*}2ike^{-2ik(x-ct)}$$

$$\psi_{2xy}=A_\xi\varphi_{2y}^{(1)}ike^{ik(x-ct)}+A^2\varphi_{2y}^{(2)}2ike^{2ik(x-ct)}-A^*{}_\xi\varphi_{2y}^{(1)}ike^{-ik(x-ct)}-A^{2*}\varphi_{2y}^{(2)*}2ike^{-2ik(x-ct)}$$

$$\psi_{2xx}=-k^2A_\xi\varphi_2^{(1)}e^{ik(x-ct)}-4k^2A^2\varphi_2^{(2)}e^{2ik(x-ct)}-k^2A^*{}_\xi\varphi_2^{(1)}e^{-ik(x-ct)}-4k^2A^{2*}\varphi_2^{(2)*}e^{-2ik(x-ct)}$$

$$\psi_{2y}=AA^*\varphi_{2y}^{(0)}+A_\xi\varphi_{2y}^{(1)}e^{ik(x-ct)}+A^2\varphi_{2y}^{(2)}e^{2ik(x-ct)}+AA^*\varphi_{2y}^{(0)*}+A^*{}_\xi\varphi_{2y}^{(1)*}e^{-ik(x-ct)}+A^{2*}\varphi_{2y}^{(2)*}e^{-2ik(x-ct)}$$

$$\psi_{2yy}=AA^*\varphi_{2yy}^{(0)}+A_\xi\varphi_{2yy}^{(1)}e^{ik(x-ct)}+A^2\varphi_{2yy}^{(2)}e^{2ik(x-ct)}+AA^*\varphi_{2yy}^{(0)*}+A^*{}_\xi\varphi_{2yy}^{(1)*}e^{-ik(x-ct)}+A^{2*}\varphi_{2yy}^{(2)*}e^{-2ik(x-ct)}$$

$$\psi_{2xxx}=-ik^3A_\xi\varphi_2^{(1)}e^{ik(x-ct)}-8ik^3A^2\varphi_2^{(2)}e^{2ik(x-ct)}-ik^3A^*{}_\xi\varphi_2^{(1)*}e^{-ik(x-ct)}-8ik^3A^{2*}\varphi_2^{(2)*}e^{-2ik(x-ct)}$$

$$\psi_{2yyx}=A_\xi\varphi_{2yy}^{(1)}ike^{ik(x-ct)}+2ikA^2\varphi_{2yy}^{(2)}e^{2ik(x-ct)}-A^*{}_\xi\varphi_{2yy}^{(1)*}ike^{-ik(x-ct)}-A^{2*}\varphi_{2yy}^{(2)*}2ike^{-2ik(x-ct)}$$

$$\psi_{2xxt}=ik^3cA_\xi\varphi_2^{(1)}e^{ik(x-ct)}+8ik^3cA^2\varphi_2^{(2)}e^{2ik(x-ct)}-ik^3A^*{}_\xi\varphi_2^{(1)}e^{-ik(x-ct)}-8ik^3cA^{2*}\varphi_2^{(2)*}e^{-2ik(x-ct)}$$

$$\psi_{2yyt}=-ikcA_\xi\varphi_{2yy}^{(1)}e^{ik(x-ct)}-2ikcA^2\varphi_{2yy}^{(2)}e^{2ik(x-ct)}+ikcA^*{}_\xi\varphi_{2yy}^{(1)*}e^{-ik(x-ct)}+2ikcA^{2*}\varphi_{2yy}^{(2)*}e^{-2ik(x-ct)}.$$

Substituting the derivatives on the left-hand side of the equation (2.34) we obtain:

$$A_{\xi}\mathrm{e}^{ik(x-ct)}\begin{pmatrix} ik^3c\varphi_2^{(1)} - ikc\varphi_{2yy}^{(1)} - ik^3U\varphi_2^{(1)} + ikU\varphi_{2yy}^{(1)} - ikU_{yy}\varphi_2^{(1)} \\ + ik\dfrac{2}{R}U\varphi_{2y}^{(1)} - \dfrac{c_{\mathrm{f}}}{2h}\left(Uk^2\varphi_2^{(1)} - 2U_y\varphi_{2y}^{(1)} - 2U\varphi_{2yy}^{(1)}\right) \end{pmatrix}$$

$$+ A^2\mathrm{e}^{2ik(x-ct)}\begin{pmatrix} 8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)} \\ + \dfrac{2}{R}2ikU\varphi_{2y}^{(2)} - \dfrac{c_{\mathrm{f}}}{2h}\left(4k^2U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right) \end{pmatrix}$$

$$+ A^*{}_{\xi}\mathrm{e}^{-ik(x-ct)}\begin{pmatrix} -ik^3c\varphi_2^{(1)*} + ikc\varphi_{2yy}^{(1)*} - ik^3U\varphi_2^{(1)*} - ikU\varphi_{2yy}^{(1)*} - ikU_{yy}\varphi_2^{(1)*} \\ -ik\dfrac{2}{R}U\varphi_{2y}^{(1)*} - \dfrac{c_{\mathrm{f}}}{2h}\left(Uk^2\varphi_2^{(1)*} - 2U_y\varphi_{2y}^{(1)*} - 2U\varphi_{2yy}^{(1)*}\right) \end{pmatrix}$$

$$+ A^{2*}\mathrm{e}^{-2ik(x-ct)}\begin{pmatrix} -8ik^3c\varphi_2^{(2)*} + 2ikc\varphi_{2yy}^{(2)*} - 8ik^3U\varphi_2^{(2)*} - 2ikU\varphi_{2yy}^{(2)*} + 2ikU_{yy}\varphi_2^{(2)*} \\ -\dfrac{2}{R}2ikU\varphi_{2y}^{(2)*} - \dfrac{c_{\mathrm{f}}}{2h}\left(4k^2U\varphi_2^{(2)*} - 2U_y\varphi_{2y}^{(2)*} - 2U\varphi_{2yy}^{(2)*}\right) \end{pmatrix}$$

$$+ \frac{c_{\mathrm{f}}}{2h}AA^*\left(2U_y\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right) + 2U\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{(0)*}\right)\right)$$

Terms proportional to AA^*:

$$\frac{c_{\mathrm{f}}}{2h}\left(2U_y\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right) + 2U\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{(0)*}\right)\right). \tag{2.42}$$

Terms proportional to $A_{\xi} \cdot \mathrm{e}^{ik(x-ct)}$:

$$\begin{aligned} &ik^3c\varphi_2^{(1)} - ikc\varphi_{2yy}^{(1)} - ik^3U\varphi_2^{(1)} + ikU\varphi_{2yy}^{(1)} - ikU_{yy}\varphi_2^{(1)} + ik\frac{2}{R}U\varphi_{2y}^{(1)} \\ &- \frac{c_{\mathrm{f}}}{2h}\left(Uk^2\varphi_2^{(1)} - 2U_y\varphi_{2y}^{(1)} - 2U\varphi_{2yy}^{(1)}\right). \end{aligned} \tag{2.43}$$

Terms proportional to $A^2 \cdot \mathrm{e}^{2ik(x-ct)}$:

$$\begin{aligned} &8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)} + \frac{2}{R}2ikU\varphi_{2y}^{(2)} \\ &- \frac{c_{\mathrm{f}}}{2h}\left(4k^2U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right). \end{aligned} \tag{2.44}$$

Collecting the terms proportional to AA^* on the left-hand side of the equation (2.34) and using (2.38) we obtain the equation for $\varphi_2^{(0)}$**:**

$$\frac{c_{\mathrm{f}}}{2h}\left(2U_y\left(\varphi_{2y}^{(0)}+\varphi_{2y}^{(0)*}\right)+2U\left(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{(0)*}\right)\right)$$
$$=ik\left(\varphi_{1y}\varphi_{1yy}^*-\varphi_{1y}^*\varphi_{1yy}+\varphi_1\varphi_{1yyy}^*-\varphi_1^*\varphi_{1yyy}\right)-\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}k^2\varphi_1\varphi_{1y}^*+k^2\varphi_1^*\varphi_{1y}\\+2\varphi_{1y}\varphi_{1yy}^*+2\varphi_{1y}^*\varphi_{1yy}\end{pmatrix}$$

Since $\varphi_{2y}^{(0)}$ is real we have $\varphi_2^{(0)}=\varphi_2^{(0)*}$. Using $S=\frac{c_{\mathrm{f}}\delta_*}{h}$ the equation for $\varphi_2^{(0)}$ is transformed to the form:

$$\begin{aligned}&4S\left(U_y\varphi_{2y}^{(0)}+U\varphi_{2yy}^{(0)}\right)=ik\left(\varphi_{1y}\varphi_{1yy}^*-\varphi_{1y}^*\varphi_{1yy}+\varphi_1\varphi_{1yyy}^*-\varphi_1^*\varphi_{1yyy}\right)\\&-\frac{S}{2}\left(k^2\varphi_1\varphi_{1y}^*+k^2\varphi_1^*\varphi_{1y}+2\varphi_{1y}\varphi_{1yy}^*+2\varphi_{1y}^*\varphi_{1yy}\right)\end{aligned}\qquad(2.45)$$

The boundary conditions are

$$\varphi_2^{(0)}(\pm\infty)=0.\qquad(2.46)$$

Similarly, collecting the terms proportional to $\mathrm{e}^{ik(x-ct)}$ on the left-hand side of the equation (2.34) and using (2.39) we obtain the following equation for the function $\varphi_2^{(1)}$:

$$ik^3c\varphi_2^{(1)}-ikc\varphi_{2yy}^{(1)}-ik^3U\varphi_2^{(1)}+ikU\varphi_{2yy}^{(1)}-ikU_{yy}\varphi_2^{(1)}+ik\frac{2}{R}U\varphi_{2y}^{(1)}$$
$$-\frac{c_{\mathrm{f}}}{2h}\left(Uk^2\varphi_2^{(1)}-2U_y\varphi_{2y}^{(1)}-2U\varphi_{2yy}^{(1)}\right)$$
$$=\left(c_g-U\right)\varphi_{1yy}-\frac{2}{R}U\varphi_{1y}+\left(-c_gk^2-2k^2c+3Uk^2+U_{yy}-\frac{c_{\mathrm{f}}}{2h}2Uik\right)\varphi_1$$

Dividing both sides by *ik* we obtain:

$$\begin{aligned}&\left(U-c-SU\frac{i}{k}\right)\varphi_{2yy}^{(1)}+\left(\frac{2}{R}U-SU_y\frac{i}{k}\right)\varphi_{2y}^{(1)}+\left(k^2c-k^2U-U_{yy}+SU\frac{ik}{2}\right)\varphi_2^{(1)}\\&=-\frac{i}{k}\left(c-U\right)\varphi_{1yy}+\frac{2i}{Rk}U\varphi_{1y}+\left(ikc+2ikc-3ikU-\frac{i}{k}U_{yy}-SU\right)\varphi_1\end{aligned}\qquad(2.47)$$

with the boundary conditions

$$\varphi_2^{(1)}(\pm\infty)=0.\qquad(2.48)$$

Finally, collecting the terms proportional to $\mathrm{e}^{2ik(x-ct)}$ we obtain:

$$8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)} + \frac{2}{R}2ikU\varphi_{2y}^{(2)}$$
$$-\frac{c_\mathrm{f}}{2h}\left(4k^2U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right)$$
$$= ik\left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) - \frac{c_\mathrm{f}}{2h}\left(-3k^2\varphi_1\varphi_{1y} + 2\varphi_{1y}\varphi_{1yy}\right) - \frac{2}{R}ik\varphi_{1y}^2$$

Dividing both sides by *ik* the following equation for the function $\varphi_2^{(2)}$ is obtained:

$$\begin{gathered}\left(2U - 2c - SU\frac{i}{k}\right)\varphi_{2yy}^{(2)} + \left(\frac{4}{R}U - SU_y\frac{i}{k}\right)\varphi_{2y}^{(2)} + \begin{pmatrix}8ck^2 - 8k^2U \\ -2U_{yy} + 2SUik\end{pmatrix}\varphi_2^{(2)} \\ = \left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) - \frac{S}{2}\left(3ik\varphi_1\varphi_{1y} - 2\frac{i}{k}\varphi_{1y}\varphi_{1yy}\right) - \frac{2}{R}\varphi_{1y}^2\end{gathered} \tag{2.49}$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm\infty) = 0. \tag{2.50}$$

Comparing (2.47) and (2.36) one can see that the left-hand side of (2.47) is exactly the same as the left-hand side of (2.36) if $\varphi_2^{(1)}(y)$ is replaced by $\varphi_1(y)$. Thus, (2.47) is resonantly forced and solvability condition should be applied at this stage to guarantee the existence of the solution. Using the Fredholm's alternative [69] we conclude that equation (2.47) has a solution if and only if the left-hand side of (2.47) is orthogonal to all eigenfunctions of the corresponding homogeneous adjoint problem.

The adjoint operator L^a and adjoint eigenfunction φ_1^a are defined by the relation

$$\int_{-\infty}^{+\infty}\varphi_1^\mathrm{a}\cdot L\varphi_1 dy = \int_{-\infty}^{+\infty}\varphi_1\cdot L^\mathrm{a}\varphi_1^\mathrm{a} dy. \tag{2.51}$$

The left-hand side of (2.51) is equal to zero since $L\varphi_1 = 0$. Thus, the adjoint equation is defined by the formula

$$L^\mathrm{a}\varphi_1^\mathrm{a} = 0\,. \tag{2.52}$$

Integrating the left-hand side of (2.51) by parts and using the boundary conditions (2.37) we obtain:

$$\int_{-\infty}^{\infty} \varphi_1^{a}\left(\varphi_1''\left(U - c - \frac{iSU}{k} \right) + \varphi_1'\left(\frac{2}{R}U - \frac{iSU_y}{k} \right) + \varphi_1\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2} \right) \right) dy$$

$$= \int_{-\infty}^{\infty} \varphi_1\left(\varphi_1^{a\,\prime\prime}\left(U - c - \frac{iSU}{k} \right) + 2\varphi_1^{a\,\prime}\left(U_y - \frac{iSU_y}{k} \right) + \varphi_1^{a}\left(U_{yy} - \frac{iSU_{yy}}{k} \right) \right) dy$$

$$- \int_{-\infty}^{\infty} \varphi_1\varphi_1^{a\,\prime}\left(\frac{2}{R}U - \frac{iSU_y}{k} \right) + \varphi_1^{a}\left(\frac{2}{R}U_y - \frac{iSU_{yy}}{k} \right) dy$$

$$+ \int_{-\infty}^{\infty} \varphi_1\varphi_1^{a}\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2} \right) dy$$

$$= \int_{-\infty}^{\infty} \varphi_1 \left(\begin{array}{l} \varphi_1^{a\,\prime\prime}\left(U - c - \frac{iSU}{k} \right) + \varphi_1^{a\,\prime}\left(2U_y - \frac{iSU_y}{k} - \frac{2}{R}U \right) \\ + \varphi_1^{a}\left(k^2c - k^2U - \frac{2}{R}U_y + \frac{ikSU}{2} \right) \end{array} \right) dy = \int_{-\infty}^{\infty} \varphi_1 \cdot L^{a}\varphi_1^{a} dy$$

Hence, the adjoint operator is

$$\begin{array}{l} L^{a}\varphi_1^{a} \equiv \varphi_{1yy}^{a}\left(U - c - \frac{iSU}{k} \right) + \varphi_{1y}^{a}\left(2U_y - \frac{iSU_y}{k} - \frac{2}{R}U \right) \\ + \varphi_1^{a}\left(k^2c - k^2U + \frac{ikSU}{2} - \frac{2}{R}U_y \right) = 0 \end{array} \tag{2.53}$$

The boundary conditions are

$$\varphi_1^{a}(\pm\infty) = 0. \tag{2.54}$$

The adjoint eigenfunction φ_1^{a} is the solution of the problem (2.53), (2.54). Applying the solvability condition to (2.47) we obtain

$$\int_{-\infty}^{+\infty} \varphi_1^{a}\left(U\varphi_{1y}\left(c_g - U \right)\varphi_{1yy} - \frac{2}{R} + \left(\begin{array}{l} -k^2c_g - 2k^2c \\ +3k^2U + U_{yy} - ikUS \end{array} \right)\varphi_1 \right) dy = 0. \tag{2.55}$$

Equation (2.55) defines the group velocity:

$$c_g = \frac{\int_{-\infty}^{+\infty} \varphi_1^{a}\left(U\varphi_{1yy} + \frac{2}{R}U\varphi_{1y} + \left(2k^2c - 3k^2U - U_{yy} + ikUS \right)\varphi_1 \right) dy}{\int_{-\infty}^{+\infty} \varphi_1^{a}\left(\varphi_{1yy} - k^2 \right) dy} \tag{2.56}$$

or

$$c_g = \frac{\eta_1}{\eta}, \tag{2.57}$$

where

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} (\varphi_{1yy} - k^2 \varphi_1) dy, \tag{2.58}$$

$$\eta_1 = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \left(U\varphi_{1yy} + \frac{2}{R} U\varphi_{1y} + (2k^2 c - 3k^2 u_0 - U_{yy} + ikUS)\varphi_1 \right) dy. \tag{2.59}$$

Solving three boundary value problems (2.45) - (2.46), (2.47) - (2.48) and (2.49) - (2.50) numerically we obtain the functions $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$. The function ψ_2 (the second order correction) is then given by (2.41).

Let us consider the solution at the third order in ε. Equation (2.35) also has a solution if and only if the right-hand side of (2.35) is orthogonal to all eigenfunctions φ_1^{a} of the corresponding homogeneous adjoint problem (2.53), (2.54). Applying the solvability condition to (2.35) we obtain:

$$\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} L \psi_3 dy = 0$$

$$\int_{-\infty}^{\infty} \varphi_1^{\mathrm{a}} \left(\begin{array}{l} c_g(\psi_{2xx\xi} + \psi_{2yy\xi}) - \\ -\psi_{1xx\tau} - 2\psi_{2x\xi\ t} + 2c_g\psi_{1x\xi\xi} - \psi_{1\xi\xi\ t} - \\ -\psi_{1yy\tau} - 3U\psi_{2xx\xi} - 3U\psi_{1x\xi\xi} - \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \\ -\psi_{2y}\psi_{1xxx} - \psi_{2y}\psi_{1yyx} - \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi\ yy} - U\psi_{2\xi\ yy} + \\ +\psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} + \\ +\psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}U_{yy} - \\ -\dfrac{c_{\mathrm{f}}}{2h} \left(\begin{array}{l} \psi_{1xx}\psi_{2y} + \dfrac{3\psi_{1xx}\psi_{1x}^2}{2U} + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + \\ +2U\psi_{2x\xi} + U\psi_{1\xi\xi} + 2\psi_{1yy}\psi_{2y} + 2\psi_{2yy}\psi_{1y} - \\ -U\psi_{1xx} - 2U_y\psi_{1y} - 2U\psi_{1yy} + 2\psi_{1x}\psi_{2xy} + \\ 2\psi_{1x}\psi_{1\xi\ y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy} \end{array} \right) - \\ -\dfrac{2}{R}\left(U\psi_{2\xi\ y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi\ y} + \psi_{1xy}\psi_{2y} \right) \end{array} \right) dy = 0 \tag{2.60}$$

The derivatives of (2.22) and (2.41) are:

$$\psi_{1x} = A\varphi_1 ik e^{ik(x-ct)} - ikA^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1\xi} = A_\xi\varphi_1 e^{ik_c(x-ct)} + A_\xi^*\varphi_1^* e^{-ik_c(x-ct)}$$

$$\psi_{1\xi\ y} = A_\xi\varphi_{1y} e^{ik_c(x-ct)} + A_\xi^*\varphi_{1y}^* e^{-ik_c(x-ct)}$$

$$\psi_{1xx} = -k^2 A\varphi_1 e^{ik(x-ct)} - k^2 A^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xx\xi} = -k^2 A_\xi\varphi_1 e^{ik(x-ct)} - k^2 A_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xxt} = ik^3 cA\varphi_1 e^{ik(x-ct)} - ik^3 cA^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xxy} = -k^2 A\varphi_{1y} e^{ik(x-ct)} - k^2 A^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1xyy} = A\varphi_{1yy} ik e^{ik(x-ct)} - ikA^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1xxx} = -ik^3 A\varphi_1 e^{ik(x-ct)} + ik^3 A^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1y} = A\varphi_{1y} e^{ik(x-ct)} + A^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1yy} = A\varphi_{1yy} e^{ik(x-ct)} + A^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1yy\xi} = A_\xi\varphi_{1yy} e^{ik(x-ct)} + A_\xi^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1xyy} = A\varphi_{1yy} ik e^{ik(x-ct)} - ikA^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1yyy} = A\varphi_{1yyy} e^{ik(x-ct)} + A^*\varphi_{1yyy}^* e^{-ik(x-ct)}$$

$$\psi_{1xy} = ikA\varphi_{1y} e^{ik(x-ct)} - ikA^*\varphi_{1y}^* e^{-ik(x-ct)}$$

$$\psi_{1x\xi} = ikA_\xi\varphi_1 e^{ik(x-ct)} - ikA_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1x\xi\ t} = k^2 cA_\xi\varphi_1 e^{ik(x-ct)} + k^2 cA_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xx\tau} = -k^2 A_\tau\varphi_1 e^{ik(x-ct)} - k^2 A_\tau^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1yy\tau} = A_\tau\varphi_{1yy} e^{ik(x-ct)} + A_\tau^*\varphi_{1yy}^* e^{-ik(x-ct)}$$

$$\psi_{1x\xi\xi} = ikA_{\xi\xi}\varphi_1 e^{ik(x-ct)} - ikA_{\xi\xi}^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1xx\xi} = -k^2 A_\xi\varphi_1 e^{ik(x-ct)} - k^2 A_\xi^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{1\xi\xi\ t} = -ikcA_{\xi\xi}\varphi_1 e^{ik(x-ct)} + ikcA_{\xi\xi}^*\varphi_1^* e^{-ik(x-ct)}$$

$$\psi_{2x} = A_\xi\varphi_2^{(1)} ik e^{ik(x-ct)} + A^2\varphi_2^{(2)} 2ik e^{2ik(x-ct)} - A_\xi^*\varphi_2^{(1)} ik e^{-ik(x-ct)} - A^{2*}\varphi_2^{(2)*} 2ik e^{-2ik(x-ct)}$$

$$\psi_{2xy} = A_\xi\varphi_{2y}^{(1)} ik e^{ik(x-ct)} + A^2\varphi_{2y}^{(2)} 2ik e^{2ik(x-ct)} - A_\xi^*\varphi_{2y}^{(1)} ik e^{-ik(x-ct)} - A^{2*}\varphi_{2y}^{(2)*} 2ik e^{-2ik(x-ct)}$$

$$\psi_{2xx} = -k^2 A_\xi\varphi_2^{(1)} e^{ik(x-ct)} - 4k^2 A^2\varphi_2^{(2)} e^{2ik(x-ct)} - k^2 A_\xi^*\varphi_2^{(1)} e^{-ik(x-ct)} - 4k^2 A^{2*}\varphi_2^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2xxx} = -ik^3 A_\xi\varphi_2^{(1)} e^{ik(x-ct)} - 8ik^3 A^2\varphi_2^{(2)} e^{2ik(x-ct)} - ik^3 A_\xi^*\varphi_2^{(1)*} e^{-ik(x-ct)} - 8ik^3 A^{2*}\varphi_2^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2yyx} = A_\xi\varphi_{2yy}^{(1)} ik e^{ik(x-ct)} + 2ikA^2\varphi_{2yy}^{(2)} e^{2ik(x-ct)} - A_\xi^*\varphi_{2yy}^{(1)*} ik e^{-ik(x-ct)} - A^{2*}\varphi_{2yy}^{(2)*} 2ik e^{-2ik(x-ct)}$$

$$\psi_{2xxt} = ik^3 cA_\xi\varphi_2^{(1)} e^{ik(x-ct)} + 8ik^3 cA^2\varphi_2^{(2)} e^{2ik(x-ct)} - ik^3 A_\xi^*\varphi_2^{(1)} e^{-ik(x-ct)} - 8ik^3 cA^{2*}\varphi_2^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2y} = AA^*\varphi_{2y}^{(0)} + A_\xi\varphi_{2y}^{(1)} e^{ik(x-ct)} + A^2\varphi_{2y}^{(2)} e^{2ik(x-ct)} + AA^*\varphi_{2y}^{(0)*} + A_\xi^*\varphi_{2y}^{(1)*} e^{-ik(x-ct)} + A^{2*}\varphi_{2y}^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2yy} = AA^*\varphi_{2yy}^{(0)} + A_\xi\varphi_{2yy}^{(1)} e^{ik(x-ct)} + A^2\varphi_{2yy}^{(2)} e^{2ik(x-ct)} + AA^*\varphi_{2yy}^{(0)*} + A_\xi^*\varphi_{2yy}^{(1)*} e^{-ik(x-ct)} + A^{2*}\varphi_{2yy}^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2yyt} = -ikcA_\xi\varphi_{2yy}^{(1)} e^{ik(x-ct)} - 2ikcA^2\varphi_{2yy}^{(2)} e^{2ik(x-ct)} + ikcA_\xi^*\varphi_{2yy}^{(1)*} e^{-ik(x-ct)} + 2ikcA^{2*}\varphi_{2yy}^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2xx\xi} = -k^2 A_{\xi\xi}\varphi_2^{(1)} e^{ik(x-ct)} - 8k^2 AA_\xi\varphi_2^{(2)} e^{2ik(x-ct)} - k^2 A_{\xi\xi}^*\varphi_2^{(1)} e^{-ik(x-ct)} - 8k^2 A^* A_\xi^*\varphi_2^{(2)*} e^{-2ik(x-ct)}$$

$$\psi_{2yy\xi} = A_\xi A^*\varphi_{2yy}^{(0)} + AA_\xi^*\varphi_{2yy}^{(0)} + A_{\xi\xi}\varphi_{2yy}^{(1)} e^{ik(x-ct)} + 2A_\xi\varphi_{2yy}^{(2)} e^{2ik(x-ct)} + A_\xi A^*\varphi_{2yy}^{(0)*} + AA_\xi^*\varphi_{2yy}^{(0)*} + A_{\xi\xi}^*\varphi_{2yy}^{(1)*} e^{-ik(x-ct)} + 2A^* A_\xi^*\varphi_{2yy}^{(2)*} e^{-2ik(x-ct)}$$

Substituting the derivatives into the right-hand side of (2.35) we obtain:

$$
\begin{aligned}
& c_g \left(\begin{array}{l}
-k^2 A_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} - 8k^2 A A_{\xi} \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\
-k^2 A^*{}_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{-ik(x-ct)} - 8k^2 A^* A^*{}_{\xi} \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\
+ A_{\xi} A^* \varphi_{2yy}^{(0)} + A A^*{}_{\xi} \varphi_{2yy}^{(0)} + A_{\xi\xi} \varphi_{2yy}^{(1)} \mathrm{e}^{ik(x-ct)} + 2 A_{\xi} \varphi_{2yy}^{(2)} \mathrm{e}^{2ik(x-ct)} \\
+ A_{\xi} A^* \varphi_{2yy}^{(0)*} + A A^*{}_{\xi} \varphi_{2yy}^{(0)*} + A^*{}_{\xi\xi} \varphi_{2yy}^{(1)*} \mathrm{e}^{-ik(x-ct)} + 2 A^* A^*{}_{\xi} \varphi_{2yy}^{(2)*} \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \\
& - \left(-k^2 A_{\tau} \varphi_1 \mathrm{e}^{ik(x-ct)} - k^2 A_{\tau}{}^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} + A_{\tau} \varphi_{1yy} \mathrm{e}^{ik(x-ct)} + A_{\tau}{}^* \varphi_{1yy}^* \mathrm{e}^{-ik(x-ct)} \right) \\
& - 2 \left(\begin{array}{l}
A_{\xi\xi} \varphi_2^{(1)} k^2 c \mathrm{e}^{ik(x-ct)} + 8 A A_{\xi} \varphi_2^{(2)} k^2 c \mathrm{e}^{2ik(x-ct)} \\
+ A^*{}_{\xi\xi} \varphi_2^{(1)} k^2 \mathrm{e}^{-ik(x-ct)} + 8 A^* A_{\xi}{}^* \varphi_2^{(2)*} k^2 \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \\
& + 2 c_g \left(ik A_{\xi\xi} \varphi_1 \mathrm{e}^{ik(x-ct)} - ik A^*{}_{\xi\xi} \varphi_1^* \mathrm{e}^{-ik(x-ct)} \right) - 3 \left(ik A_{\xi\xi} \varphi_1 \mathrm{e}^{ik(x-ct)} - ik A^*{}_{\xi\xi} \varphi_1^* \mathrm{e}^{-ik(x-ct)} \right) \\
& - \left(-ikc A_{\xi\xi} \varphi_1 \mathrm{e}^{ik_c(x-ct)} + ikc A^*{}_{\xi\xi} \varphi_1{}^* \mathrm{e}^{-ik_c(x-ct)} \right) \\
& - 3U \left(\begin{array}{l}
-k^2 A_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} - 8k^2 A A_{\xi} \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\
-k^2 A^*{}_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{-ik(x-ct)} - 8k^2 A^* A^*{}_{\xi} \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \\
& - U \left(\begin{array}{l}
A_{\xi} A^* \varphi_{2yy}^{(0)} + A A^*{}_{\xi} \varphi_{2yy}^{(0)} + A_{\xi\xi} \varphi_{2yy}^{(1)} \mathrm{e}^{ik(x-ct)} + 2 A_{\xi} \varphi_{2yy}^{(2)} \mathrm{e}^{2ik(x-ct)} \\
+ A_{\xi} A^* \varphi_{2yy}^{(0)*} + A A^*{}_{\xi} \varphi_{2yy}^{(0)*} + A^*{}_{\xi\xi} \varphi_{2yy}^{(1)*} \mathrm{e}^{-ik(x-ct)} + 2 A^* A^*{}_{\xi} \varphi_{2yy}^{(2)*} \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \\
& + U_{yy} \left(\begin{array}{l}
A_{\xi} A^* \varphi_2^{(0)} + A A^*{}_{\xi} \varphi_2^{(0)} + A_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} + 2 A_{\xi} \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\
+ A_{\xi} A^* \varphi_2^{(0)*} + A A^*{}_{\xi} \varphi_2^{(0)*} + A^*{}_{\xi\xi} \varphi_2^{(1)*} \mathrm{e}^{-ik(x-ct)} + 2 A^* A^*{}_{\xi} \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \\
& - \left(A \varphi_{1y} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \right) \left(\begin{array}{l}
-ik^3 A_{\xi} \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} - 8ik^3 A^2 \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\
-ik^3 A^*{}_{\xi} \varphi_2^{(1)*} \mathrm{e}^{-ik(x-ct)} - 8ik^3 A^{2*} \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\
+ 3 \left(-k^2 A_{\xi} \varphi_1 \mathrm{e}^{ik(x-ct)} - k^2 A_{\xi}{}^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} \right) \\
+ A_{\xi} \varphi_{2yy}^{(1)} ik \mathrm{e}^{ik(x-ct)} + 2ik A^2 \varphi_{2yy}^{(2)} \mathrm{e}^{2ik(x-ct)} \\
- ik A^*{}_{\xi} \varphi_{2yy}^{(1)*} \mathrm{e}^{-ik(x-ct)} - 2ik A^{2*} \varphi_{2yy}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\
+ A_{\xi} \varphi_{1yy} \mathrm{e}^{ik(x-ct)} + A^*{}_{\xi} \varphi_{1yy}^* \mathrm{e}^{-ik(x-ct)}
\end{array} \right) \\
& - \left(\begin{array}{l}
A A^* \varphi_{2y}^{(0)} + A_{\xi} \varphi_{2y}^{(1)} \mathrm{e}^{ik(x-ct)} \\
+ A^2 \varphi_{2y}^{(2)} \mathrm{e}^{2ik(x-ct)} + A A^* \varphi_{2y}^{(0)*} \\
+ A^*{}_{\xi} \varphi_{2y}^{(1)*} \mathrm{e}^{-ik(x-ct)} + A^{2*} \varphi_{2y}^{(2)*} \mathrm{e}^{-2ik(x-ct)}
\end{array} \right) \left(\begin{array}{l}
-ik^3 A \varphi_1 \mathrm{e}^{ik(x-ct)} + ik^3 A^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} \\
+ A \varphi_{1yy} ik \mathrm{e}^{ik(x-ct)} - ik A^* \varphi_{1yy}^* \mathrm{e}^{-ik(x-ct)}
\end{array} \right) \\
& + \left(\begin{array}{l}
A \varphi_1 ik \mathrm{e}^{ik(x-ct)} \\
- ik A^* \varphi_1^* \mathrm{e}^{-ik(x-ct)}
\end{array} \right) \left(\begin{array}{l}
-k^2 A_{\xi} \varphi_{2y}^{(1)} \mathrm{e}^{ik(x-ct)} - 4k^2 A^2 \varphi_{2y}^{(2)} \mathrm{e}^{2ik(x-ct)} \\
-k^2 A^*{}_{\xi} \varphi_{2y}^{(1)} \mathrm{e}^{-ik(x-ct)} - 4k^2 A^{2*} \varphi_{2y}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\
+ A A^* \varphi_{2yy}^{(0)} + A_{\xi} \varphi_{2yy}^{(1)} \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2yy}^{(2)} \mathrm{e}^{2ik(x-ct)} \\
+ A A^* \varphi_{2yy}^{(0)*} + A^*{}_{\xi} \varphi_{2yy}^{(1)*} \mathrm{e}^{-ik(x-ct)} + A^{2*} \varphi_{2yy}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\
+ 2 \left(ik A_{\xi} \varphi_{1y} \mathrm{e}^{ik(x-ct)} - ik A_{\xi}{}^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \right)
\end{array} \right)
\end{aligned}
$$

$$
+\begin{pmatrix} A_\xi \varphi_2^{(1)} ik \mathrm{e}^{ik(x-ct)} + A^2 \varphi_2^{(2)} 2ik \mathrm{e}^{2ik(x-ct)} \\ - A^*{}_\xi \varphi_2^{(1)} ik \mathrm{e}^{-ik(x-ct)} - A^{2*} \varphi_2^{(2)*} 2ik \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \begin{pmatrix} -k^2 A \varphi_{1y} \mathrm{e}^{ik(x-ct)} - k^2 A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \\ + A \varphi_{1yyy} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1yyy}^* \mathrm{e}^{-ik(x-ct)} \end{pmatrix}
$$

$$
+\left(A_\xi \varphi_1 \mathrm{e}^{ik_c(x-ct)} + A_\xi{}^* \varphi_1{}^* \mathrm{e}^{-ik_c(x-ct)} \right) \begin{pmatrix} -k^2 A \varphi_{1y} \mathrm{e}^{ik(x-ct)} - k^2 A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \\ + A \varphi_{1yyy} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1yyy}^* \mathrm{e}^{-ik(x-ct)} \end{pmatrix}
$$

$$
-\frac{2}{R} \left(\begin{array}{l}
U \begin{pmatrix} A_\xi A^* \varphi_{2y}^{(0)} + A A^*{}_\xi \varphi_{2y}^{(0)} + A_{\xi\xi} \varphi_{2y}^{(1)} \mathrm{e}^{ik(x-ct)} + 2 A_\xi \varphi_{2y}^{(2)} \mathrm{e}^{2ik(x-ct)} \\ + A_\xi A^* \varphi_{2y}^{(0)*} + A A^*{}_\xi \varphi_{2y}^{(0)*} + A^*{}_{\xi\xi} \varphi_{2y}^{(1)*} \mathrm{e}^{-ik(x-ct)} + 2 A^* A^*{}_\xi \varphi_{2y}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \\
+ \left(A \varphi_{1y} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \right) \begin{pmatrix} A_\xi \varphi_{2y}^{(1)} ik \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2y}^{(2)} 2ik \mathrm{e}^{2ik(x-ct)} + \psi_{1\xi\ y} \\ - A^*{}_\xi \varphi_{2y}^{(1)} ik \mathrm{e}^{-ik(x-ct)} - A^{2*} \varphi_{2y}^{(2)*} 2ik \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \\
+ \begin{pmatrix} A A^* \varphi_{2y}^{(0)} + A_\xi \varphi_{2y}^{(1)} \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2y}^{(2)} \mathrm{e}^{2ik(x-ct)} \\ + A A^* \varphi_{2y}^{(0)*} + A^*{}_\xi \varphi_{2y}^{(1)*} \mathrm{e}^{-ik(x-ct)} \\ + A^{2*} \varphi_{2y}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \left(ikA \varphi_{1y} \mathrm{e}^{ik(x-ct)} - ikA^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \right)
\end{array} \right)
$$

$$
-\frac{c_{\mathrm{f}}}{2h} \left(\begin{array}{l}
2 \begin{pmatrix} A \varphi_1 ik \mathrm{e}^{ik(x-ct)} - \\ - ikA^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} \end{pmatrix} \begin{pmatrix} A_\xi \varphi_{2y}^{(1)} ik \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2y}^{(2)} 2ik \mathrm{e}^{2ik(x-ct)} \\ - A^*{}_\xi \varphi_{2y}^{(1)} ik \mathrm{e}^{-ik(x-ct)} - A^{2*} \varphi_{2y}^{(2)*} 2ik \mathrm{e}^{-2ik(x-ct)} + \psi_{1\xi\ y} \end{pmatrix} \\
+ \begin{pmatrix} A \varphi_{1y} \mathrm{e}^{ik(x-ct)} + \\ + A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \end{pmatrix} \left(\begin{array}{l} -k^2 A_\xi \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} - 4k^2 A^2 \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\ -k^2 A^*{}_\xi \varphi_2^{(1)} \mathrm{e}^{-ik(x-ct)} - 4k^2 A^{2*} \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)} \\ + 2 \begin{pmatrix} A A^* \varphi_{2yy}^{(0)} + A_\xi \varphi_{2yy}^{(1)} \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2yy}^{(2)} \mathrm{e}^{2ik(x-ct)} \\ + A A^* \varphi_{2yy}^{(0)*} + A^*{}_\xi \varphi_{2yy}^{(1)*} \mathrm{e}^{-ik(x-ct)} + A^{2*} \varphi_{2yy}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \\ + 2 \left(A_\xi \varphi_1 ik \mathrm{e}^{ik(x-ct)} - ikA^*{}_\xi \varphi_1^* \mathrm{e}^{-ik(x-ct)} \right) \end{array} \right) \\
+ \begin{pmatrix} A A^* \varphi_{2y}^{(0)} + A_\xi \varphi_{2y}^{(1)} \mathrm{e}^{ik(x-ct)} + A^2 \varphi_{2y}^{(2)} \mathrm{e}^{2ik(x-ct)} + \\ + A A^* \varphi_{2y}^{(0)*} + A^*{}_\xi \varphi_{2y}^{(1)*} \mathrm{e}^{-ik(x-ct)} + A^{2*} \varphi_{2y}^{(2)*} \mathrm{e}^{-2ik(x-ct)} \end{pmatrix} \begin{pmatrix} -k^2 A \varphi_1 \mathrm{e}^{ik(x-ct)} \\ -k^2 A^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} + 2\psi_{1yy} \end{pmatrix} \\
+ 2 \begin{pmatrix} ikA \varphi_{1y} \mathrm{e}^{ik(x-ct)} - \\ - ikA^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \end{pmatrix} \begin{pmatrix} A_\xi \varphi_2^{(1)} ik \mathrm{e}^{ik(x-ct)} + A^2 \varphi_2^{(2)} 2ik \mathrm{e}^{2ik(x-ct)} \\ - A^*{}_\xi \varphi_2^{(1)} ik \mathrm{e}^{-ik(x-ct)} - A^{2*} \varphi_2^{(2)*} 2ik \mathrm{e}^{-2ik(x-ct)} + \psi_{1\xi} \end{pmatrix} \\
+ \dfrac{3\left(-k^2 A \varphi_1 \mathrm{e}^{ik(x-ct)} - k^2 A^* \varphi_1^* \mathrm{e}^{-ik(x-ct)}\right)\left(A \varphi_1 ik \mathrm{e}^{ik(x-ct)} - ikA^* \varphi_1^* \mathrm{e}^{-ik(x-ct)}\right)^2}{2U} - \\
- 2U_y \left(A \varphi_{1y} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1y}^* \mathrm{e}^{-ik(x-ct)} \right) - 2U \left(A \varphi_{1yy} \mathrm{e}^{ik(x-ct)} + A^* \varphi_{1yy}^* \mathrm{e}^{-ik(x-ct)} \right) \\
- U \left(-k^2 A \varphi_1 \mathrm{e}^{ik(x-ct)} - k^2 A^* \varphi_1^* \mathrm{e}^{-ik(x-ct)} \right) + U \left(A_\xi \varphi_1 \mathrm{e}^{ik(x-ct)} + A_\xi{}^* \varphi_1{}^* \mathrm{e}^{-ik(x-ct)} \right) \\
+ 2U \begin{pmatrix} ikA_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{ik(x-ct)} + 4ikA A_\xi \varphi_2^{(2)} \mathrm{e}^{2ik(x-ct)} \\ - ikA^*{}_{\xi\xi} \varphi_2^{(1)} \mathrm{e}^{-ik(x-ct)} - 4ikA^* A^*{}_\xi \varphi_2^{(2)*} \mathrm{e}^{-2ik(x-ct)} \end{pmatrix}
\end{array} \right)
$$

Collect the terms proportional to $A_\tau \cdot \mathrm{e}^{ik(x-ct)}$

$$
-\psi_{1xx\tau} - \psi_{1yy\tau} \Rightarrow -\left(\varphi_{1yy} - k^2 \varphi_1 \right) \tag{2.61}
$$

Collect the terms proportional to $A \cdot \mathrm{e}^{ik(x-ct)}$

$$-\frac{c_{\mathrm{f}}}{2h}\left(-2U_y\psi_{1y}-2U\psi_{1yy}+U\psi_{1xx}\right)\Rightarrow-\frac{c_{\mathrm{f}}}{2h}\left(2U_y\varphi_{1y}+2U\varphi_{1yy}-Uk^2\varphi_1\right) \quad (2.62)$$

Collect the terms proportional to $A_{\xi\xi} \cdot \mathrm{e}^{ik(x-ct)}$

$$\begin{aligned}&c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-2\psi_{2x\xi\ t}+2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}-3U\psi_{2xx\xi}-3U\psi_{1x\xi\xi}\\&-U\psi_{2\xi\ yy}+\psi_{2\xi}U_{yy}-\frac{c_{\mathrm{f}}}{2h}\left(2U\psi_{2x\xi}+U\psi_{1\xi\xi}\right)-\frac{2}{R}U\psi_{2\xi\ y}\Rightarrow\end{aligned}$$

$$\begin{aligned}&c_g\left(-k^2\varphi_2^{(1)}+\varphi_{2yy}^{(1)}\right)-2k^2c\varphi_2^{(1)}+2c_gik\varphi_1+ikc\varphi_1+3Uk^2\varphi_2^{(1)}-3Uik\varphi_1\\&-U\varphi_{2yy}^{(1)}+U_{yy}\varphi_2^{(1)}-\frac{c_{\mathrm{f}}}{2h}\left(2Uik\varphi_2^{(1)}+U\varphi_1\right)-\frac{2}{R}U\varphi_{2y}^{(1)}\end{aligned} \quad (2.63)$$

Nonlinear terms proportional to $A|A|^2$:

$$\begin{aligned}&-\psi_{2y}\left(\psi_{1xxx}+\psi_{1yyx}\right)+\psi_{1x}\left(\psi_{2xxy}+2\psi_{1xy\xi}+\psi_{2yyy}\right)\\&-\psi_{1y}\left(\psi_{2xxx}+3\psi_{1xx\xi}+\psi_{2yyx}+\psi_{1\xi\ yy}\right)+\psi_{2x}\left(\psi_{1xxy}+\psi_{1yyy}\right)\\&+\psi_{1\xi}\left(\psi_{1xxy}+\psi_{1yyy}\right)-\frac{c_{\mathrm{f}}}{2h}\left(\begin{aligned}&\psi_{1xx}\psi_{2y}+\frac{3\psi_{1xx}\psi_{1x}^2}{2U}+\psi_{2xx}\psi_{1y}\\&+2\psi_{1\xi}\psi_{1y}+2\psi_{1yy}\psi_{2y}\\&+2\psi_{2yy}\psi_{1y}+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi y}\\&+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}\end{aligned}\right)\\&-\frac{2}{R}\left(\psi_{1y}\psi_{2xy}+\psi_{1y}\psi_{1\xi\ y}+\psi_{1xy}\psi_{2y}\right)\Rightarrow\end{aligned}$$

$$\begin{aligned}&6ik^3\varphi_2^{(2)}\varphi_{1y}^*-2ik\varphi_{1y}^*\varphi_{2yy}^{(2)}+3ik^3\varphi_1^*\varphi_{2y}^{(2)}+ik^3\varphi_1(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})+ik\varphi_{2y}^{(2)}\varphi_{1yy}^*\\&-ik\varphi_{1yy}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})-ik\varphi_1^*\varphi_{2yyy}^{(2)}+ik\varphi_1(\varphi_{2yyy}^{(0)}+\varphi_{2yyy}^{*(0)})+2ik\varphi_{1yyy}^*\varphi_2^{(2)}\\&-\frac{S}{2}\left(\begin{aligned}&-k^2\varphi_1(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})+3k^2\varphi_1^*\varphi_{2y}^{(2)}\\&-\frac{3k^4}{2u_0}\varphi_1^2\varphi_1^*+2\varphi_{1yy}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})\\&+2\varphi_{1yy}^*\varphi_{2y}^{(2)}+2\varphi_{1y}(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)})+2\varphi_{2yy}^{(2)}\varphi_{1y}^*\end{aligned}\right)-2\frac{ik}{R}\left(\begin{aligned}&\varphi_{2y}^{(2)}\varphi_{1y}^*\\&+\varphi_{2y}^{(0)}\varphi_{1y}\end{aligned}\right)\end{aligned} \quad (2.64)$$

The equation (2.60) is rewritten in the form:

$$\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\left(\psi_{1xx\tau}+\psi_{1yy\tau}\right)dy=\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\begin{pmatrix} c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-2\psi_{2x\xi\ t} \\ +2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}-3U\psi_{2xx\xi} \\ -3U\psi_{1x\xi\xi}-U\psi_{2\xi\ yy}+\psi_{2\xi}U_{yy} \\ -\dfrac{c_{\mathrm{f}}}{2h}\left(2U\psi_{2x\xi}+U\psi_{1\xi\xi}\right)-\dfrac{2}{R}U\psi_{2\xi\ y} \end{pmatrix}dy$$

$$-\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\frac{c_{\mathrm{f}}}{2h}\left(-2U_y\psi_{1y}-2U\psi_{1yy}+U\psi_{1xx}\right)dy$$

$$+\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\begin{pmatrix} -\psi_{1y}\psi_{2xxx}-3\psi_{1y}\psi_{1xx\xi}-\psi_{2y}\psi_{1xxx}-\psi_{2y}\psi_{1yyx}-\psi_{1y}\psi_{2yyx} \\ -\psi_{1y}\psi_{1\xi\ yy}+\psi_{2x}\psi_{1xxy}+\psi_{1\xi}\psi_{1xxy}+\psi_{1x}\psi_{2xxy}+2\psi_{1x}\psi_{1xy\xi} \\ +\psi_{1x}\psi_{2yyy}+\psi_{2x}\psi_{1yyy}+\psi_{1\xi}\psi_{1yyy} \\ -\dfrac{c_{\mathrm{f}}}{2h}\begin{pmatrix} \psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2U}+2\psi_{1x\xi}\psi_{1y}+\psi_{2xx}\psi_{1y} \\ +2\psi_{1yy}\psi_{2y}+2\psi_{2yy}\psi_{1y}+2\psi_{1x}\psi_{2xy} \\ +2\psi_{1x}\psi_{1\xi\ y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy} \end{pmatrix}-\dfrac{2}{R}\begin{pmatrix} \psi_{1y}\psi_{2xy} \\ +\psi_{1y}\psi_{1\xi\ y} \\ +\psi_{1xy}\psi_{2y} \end{pmatrix} \end{pmatrix}dy \qquad (2.65)$$

Using (2.61) - (2.64) equation (2.65) is rewritten as the amplitude evolution equation for slowly varying amplitude function $A(\xi,\tau)$ of the form:

$$\eta A_\tau=\sigma_1 A+\delta_1 A_{\xi\xi}-\mu_1|A|^2 A$$

or

$$\frac{\partial A}{\partial\tau}=\sigma A+\delta\frac{\partial^2 A}{\partial\xi^2}-\mu\,|\,A\,|^2\ A. \qquad (2.66)$$

Equation (2.66) is the complex Ginzburg-Landau equation with complex coefficients σ,δ and μ :

$$\sigma=\frac{\sigma_1}{\eta},\ \ \delta=\frac{\delta_1}{\eta},\ \ \mu=\frac{\mu_1}{\eta}. \qquad (2.67)$$

where $\sigma=\sigma_{\mathrm{r}}+i\sigma_{\mathrm{i}}$, $\delta=\delta_{\mathrm{r}}+i\delta_{\mathrm{i}}$ and $\mu=\mu_{\mathrm{r}}+i\mu_{\mathrm{i}}$ are complex coefficients which can be computed using linearized characteristics of the flow.

Coefficients σ_1, δ_1 μ_1 and η are given by:

$$\sigma_1 = \frac{S}{2}\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}}\left(-k^2 U\varphi_1 + 2U_y\varphi_{1y} + 2U\varphi_{1yy}\right)dy, \tag{2.68}$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \begin{pmatrix} \left(c_g - U\right)\varphi_{2yy}^{(1)} - 2\frac{U}{R}\varphi_{2y}^{(1)} \\ + \varphi_2^{(1)}\left(-k^2 c_g - 2k^2 c + 3k^2 U + U_{yy} - ikSU\right) \\ + \varphi_1\left(2ikc_g + ikc - 3ikU - U\frac{S}{2}\right) \end{pmatrix} dy, \tag{2.69}$$

$$\mu_1 = -\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \begin{pmatrix} 6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)} + ik^3\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) \\ - ik\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + ik\varphi_{2y}^{(2)}\varphi_{1yy}^* + ik\varphi_1\left(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}\right) \\ - ik\varphi_1^*\varphi_{2yyy}^{(2)} + 2ik\varphi_{1yyy}^*\varphi_2^{(2)} - 2\frac{ik}{R}\left(\varphi_{2y}^{(2)}\varphi_{1y}^* + \varphi_{2y}^{(0)}\varphi_{1y}\right) \\ -\frac{S}{2}\begin{pmatrix} -k^2\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 3k^2\varphi_1^*\varphi_{2y}^{(2)} - \frac{3k^4}{2u_0}\varphi_1^2\varphi_1^* \\ + 2\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 2\varphi_{1yy}^*\varphi_{2y}^{(2)} \\ + 2\varphi_{1y}\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}\right) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^* \end{pmatrix} \end{pmatrix} dy, \tag{2.70}$$

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}}(\varphi_{1yy} - k^2\varphi_1)dy\,. \tag{2.71}$$

The constant μ_{r} is known as the Landau constant in the literature. If $\mu_{\mathrm{r}} > 0$ then finite saturation of the amplitude is possible and (2.66) can be useful in analyzing the development of instability. There are many examples in fluid mechanics including rotating convective flows [54], [56] and shallow water flows [30], [46], where the constant $\mu_{\mathrm{r}} > 0$. However, for plane Poiseuille flow $\mu_{\mathrm{r}} < 0$ (see [33]) so that (2.66) is not useful at all since higher-order terms become important as well.

Formulas (2.67) represent the coefficients of equation (2.66) in terms of the characteristics of the linear stability of the flow. More precisely, in order to obtain σ, δ and μ we need to perform the following calculations:

1. Solve the linear stability problem (2.36) - (2.37) and determine the critical values of the parameters k, S, c and the corresponding eigenfunction $\varphi_1(y)$;
2. Solve the homogeneous adjoint problem (2.53) - (2.54) and determine the adjoint eigenfunction φ_1^{a};
3. Solve three boundary value problems (2.45) - (2.46), (2.47) - (2.48), (2.49) - (2.50) and determine the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$;
4. Evaluate the integrals in (2.67).

Ginzburg-Landau equation is often used to model spatio-temporal dynamics of complex flows. The reason is that (2.66) exhibits a rich variety of solutions depending on the values of the coefficients σ, δ and μ. In addition, it contains the terms representing linear growth, diffusion and nonlinearity. In many cases the Ginzburg-Landau equation is used as a phenomenological model, that is, it is assumed but not derived from the equations of motion. Experimental data are often used in such cases in order to estimate the coefficients of the equation.

In other cases the Ginzburg-Landau equation can be derived from the equations of motion (examples are given in [50], [58] and [61]). The coefficients of the equation are calculated in a closed form as integrals containing characteristics of the linearized problems.

Ginzburg-Landau equation and its properties are extensively studied in the literature (see, for example, [1] and [10]). Numerical analysis of the Ginzburg-Landau equation (see section 6) is simpler than numerical solution of the equations of motion. In addition, stability of some simple (for example, periodic) solutions of the Ginzburg-Landau equation allows researchers to simplify the analysis of spatio-temporal dynamics of complex flows in fluid mechanics.

2.4 Numerical Method for Weakly Nonlinear Stability

In this subsection we present a numerical method for the calculation of the coefficients of the Ginzburg-Landau equation. The solutions of linear stability problem (2.36)-(2.37), adjoint problem (2.53)-(2.54), boundary value problems (2.45)-(2.50) are sought in the same form (2.13), where $\varphi(r)$ represents any of the functions $\varphi_1(r)$, $\varphi_1^{\mathrm{a}}(r)$, $\varphi_2^{(0)}(r)$, $\varphi_2^{(1)}(r)$, $\varphi_2^{(2)}(r)$ (recall that $r = \dfrac{2}{\pi}\arctan y$). Using the chain rule we compute the derivatives of the first, second (2.14) and third order of φ with respect to y:

$$\frac{d^3\varphi}{dy^3}=\frac{\left(12\tan^2\frac{\pi r}{2}-4\right)}{\pi}\cos^6\frac{\pi r}{2}\frac{d\varphi}{dr}-\frac{24}{\pi^2}\sin\frac{\pi r}{2}\cos^5\frac{\pi r}{2}\frac{d^2\varphi}{dr^2}+\frac{8}{\pi^2}\cos^6\frac{\pi r}{2}\frac{d^3\varphi}{dr^3}. \quad (2.72)$$

The derivatives of the first, second (2.15) and third order of φ with respect to r are evaluated using (2.13):

$$\frac{d^3\varphi}{dr^3}=\sum_{j=0}^{N-1}a_j\left(-6T_j'(r)-6rT_j''(r)+(1-r^2)T_j'''(r)\right). \quad (2.73)$$

In order to evaluate the function $\varphi(r)$ and its derivatives up to the third order we need to compute the values of the Chebyshev polynomial $T_j(r)$ and its derivatives at the collocation points (2.13):

$$T_j'''(r_m)=\left(\frac{j-j^3}{\sin^3\frac{\pi m}{N+1}}+\frac{3j\cos^2\frac{\pi m}{N+1}}{\sin^5\frac{\pi m}{N+1}}\right)\sin\frac{m\pi j}{N+1}-\frac{3j^2\cos\frac{\pi m}{N+1}}{\sin^4\frac{\pi m}{N+1}}\cos\frac{\pi mj}{N+1}. \quad (2.74)$$

The values of $\varphi_1(r)$, its derivatives up to order two inclusive and the coefficients of equation (2.37) at the collocation points (2.13) can be evaluated using formulas (2.72) - (2.74) so that the elements of the matrices *B* and *D* (see (2.19)) can be computed and the generalized eigenvalue problem (2.19) can be solved numerically. Similar approach can be used in order to solve boundary value problems (2.45) - (2.46) and (2.49) - (2.50). System of linear algebraic equations of the form

$$Fa=G \quad (2.75)$$

is obtained in each case after discretization where $a=(a_0\,a_1...a_{N-1})^{\mathrm{T}}$. The matrix F is not singular for problems (2.45) - (2.46) and (2.49) - (2.50). Therefore, any linear equation solver can be used in order to find a. Thus, the functions $\varphi_2^{(0)}(y)$ and $\varphi_2^{(2)}(y)$ can be evaluated by means of the expansions of the form (2.13).

The same form of the expansion (2.13) is used to solve boundary value problem (2.47) - (2.48). Equation of the form (2.19) is also obtained after discretization in this case, but the matrix F is singular since the corresponding homogeneous part of (2.47) has a nontrivial solution at $S=S_{\mathrm{c}},k=k_{\mathrm{c}}$ and $c=c_{\mathrm{c}}$. Equation (2.75) is solved in this case by means of the singular value decomposition method [3]. It is known that if F is a complex $N \times N$ matrix, then there exist orthogonal $N \times N$ matrices U and V such that

$$U^{\mathrm{H}} \cdot F \cdot V = \Sigma, \tag{2.76}$$

where $\Sigma = diag(\gamma_1, \gamma_2, ..., \gamma_N)$.

Equation (2.76) is called the singular value decomposition of the matrix F and $\gamma_1, \gamma_2, ..., \gamma_N$ are the singular values of F. In our case only the last of the singular values will be equal to zero ($\gamma_1 > \gamma_2 > ... > \gamma_{N-1} > \gamma_N = 0$). Hence, the solution to (2.75) in this case can be written in the form

$$a = V \cdot \Sigma^{-1} \cdot U^{\mathrm{H}} \cdot G, \tag{2.77}$$

where the last column of V, the last row of U^H, the last column and the last row of Σ^{-1} are deleted. In component form the solution to (2.77) is

$$a = \sum_{i=1}^{N-1} \frac{U_i^{\mathrm{H}} \cdot G \cdot V_i}{\gamma_i}, \tag{2.78}$$

where U_i^{H} and V_i are vectors (columns of the matrices U^{H} and V, respectively).

Hence, the values of the function $\varphi_2^{(1)}(y)$ can be computed using formula (2.13) where the coefficients a_j are the components of the vector a in (2.78).

The final step of the computational procedure involves the calculation of integrals in (2.67). Adaptive quadrature formula described in [31] can be used to compute the integrals in (2.67).

3. LINEAR AND WEAKLY NONLINEAR INSTABILITY OF SLIGHTLY CURVED TWO-COMPONENT SHALLOW MIXING LAYERS

3.1 Linear Stability

Consider the two-dimensional shallow water equations in the presence of a small curvature under the rigid-lid assumption (1.1) – (1.3) [13], [14] and [28]. It is assumed that the carrier fluid contains small heavy particles. The assumptions that are used in the derivation of the governing equations are summarized in Section 1.2. Eliminating the pressure p and introducing the stream function $\psi(x,y,t)$ (see 2.1) system (1.1) – (1.3) can be reduced to one equation

$$\begin{aligned}(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} \\ + \frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) + B\Delta\psi = 0,\end{aligned} \tag{3.1}$$

A perturbed solution to (3.1) is sought in the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + ..., \tag{3.2}$$

where ε – a small parameter which will be defined later.

Substituting (3.2) into (3.1) and linearizing the resulting equation in the neighbourhood of the base flow we obtain (see 2.1):

$$L\psi_1 = 0, \tag{3.3}$$

where

$$\begin{aligned}L\psi \equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{0y}\psi_{yyx} - \psi_{0yyy}\psi_x \\ + \frac{c_{\mathrm{f}}}{2h}(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}) + \frac{2}{R}\psi_{0y}\psi_{xy} + B(\psi_{1xx} + \psi_{1yy})\end{aligned}$$

A hyperbolic tangent velocity profile of the form $\psi_{0y} = U(y)$

$$U(y) = \frac{U_1+U_2}{2} + \frac{U_2-U_1}{2}\tanh y \tag{3.4}$$

is often used in practice in order to represent the base flow for the case of a mixing layer. Here U_1 and U_2 are the velocities of undisturbed flow at $y=-\infty$ and $y=+\infty$, respectively.

The solution to (3.3) is sought in the form of a normal mode

$$\psi_1(x,y,t) = \varphi_1(y)\mathrm{e}^{ik(x-ct)}. \qquad (3.5)$$

Using (3.3) and (3.5) we obtain

$$L\varphi_1 = 0, \qquad (3.6)$$

where

$$L\varphi_1 \equiv \varphi_1''\left(U - c - \frac{iSU}{k} - \frac{iB}{k}\right) + \varphi_1'\left(\frac{2U}{R} - \frac{iSU_y}{k}\right)$$
$$+\varphi_1\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2} + ikB\right).$$

The boundary conditions are

$$\varphi_1(\pm\infty) = 0. \qquad (3.7)$$

Here $S = \dfrac{c_\mathrm{f} b}{h}$ – the stability parameter.

Note that (3.6), (3.7) is an eigenvalue problem (the complex eigenvalues are $c = c_\mathrm{r} + ic_\mathrm{i}$). Base flow (3.4) is said to be stable if all $c_\mathrm{i} < 0$ and unstable if at least one $c_\mathrm{i} > 0$. Marginal stability of flow (3.4) is described by the relation $c_\mathrm{i} = 0$. Problem (3.6), (3.7) is usually solved numerically (details of numerical algorithm based on collocation method are given in Chapter 2). Thus, solution of (3.6), (3.7) allows one to obtain the critical values of the parameters $S_\mathrm{c}, k_\mathrm{c}, c_\mathrm{c}$. A typical marginal stability curve for shallow water flows is a convex curve with one maximum (the coordinates of the maximum point in the $(k,S)-$ plane are $k = k_\mathrm{c}$ and $S = S_\mathrm{c}$).

3.2 Weakly Nonlinear Stability

Assume that the bed-friction number is slightly smaller than the critical value:

$$S = S_\mathrm{c}(1-\varepsilon^2). \qquad (3.8)$$

Now the role of the parameter ε in (3.2) becomes clear: it characterizes how close is the parameter S to the critical value S_c. In addition, (3.8) implies that base flow (3.4) is unstable if the bed-friction number is equal to S. However, since ε is small, the growth rate of the most unstable perturbation is also small. Hence, one can try to characterize the development of instability analytically by means of weakly nonlinear theory.

Following [62] we introduce the following "slow" variables

$$\tau = \varepsilon^2 t, \quad \xi = \varepsilon(x - c_g t), \tag{3.9}$$

where c_g is the group velocity.

The stream function ψ_1 in (3.5) is replaced by

$$\psi_1(x, y, t, \xi, \tau) = A(\xi, \tau)\varphi_1(y)\mathrm{e}^{ik(x-ct)}, \tag{3.10}$$

where $\varphi_1(y)$ is the eigenfunction of the marginally stable normal perturbation with $S = S_\mathrm{c}, k = k_\mathrm{c}$ and $c = c_\mathrm{c}$. The objective is to derive equation for the evolution of the amplitude function $A(\xi, \tau)$.

Using (3.9) we replace the derivatives with respect to x and t in (3.1) by the following expressions

$$\begin{aligned} &\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial \xi} \\ &\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g\frac{\partial}{\partial \xi} + \varepsilon^2\frac{\partial}{\partial \tau}. \end{aligned} \tag{3.11}$$

Using (3.1), (3.2), (3.11) and collecting the terms that contain ε^2 we obtain

$$\begin{aligned} &L_1\psi_2 = c_g\left(\psi_{1xx\xi} + \psi_{1yy\xi}\right) - 2\psi_{1x\xi\ t} - 3U\psi_{1xx\xi} - \psi_{1y}\psi_{1xxx} - \psi_{1y}\psi_{1yyx} \\ &- U\psi_{1\xi\ yy} + \psi_{1x}\psi_{1xxy} + \psi_{1x}\psi_{1yyy} + U_{yy}\psi_{1\xi} \\ &- \frac{S}{2}\left(\psi_{1xx}\psi_{1y} + 2U\psi_{1x\xi} + 2\psi_{1yy}\psi_{1y} - 2UU_y + 2\psi_{1x}\psi_{1xy}\right) \\ &- \frac{2}{R}\left(U\psi_{1\xi\ y} + \psi_{1y}\psi_{1xy}\right) - 2B\psi_{1x\xi}. \end{aligned} \tag{3.12}$$

Analyzing the structure of the right-hand side of (3.12) and using (3.10) we conclude that ψ_2 in (3.12) should be sought in the form

$$\psi_2 = AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)} + A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)}, \tag{3.13}$$

where A^* is the complex conjugate of A and $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y. Substituting (3.13) into (3.12) and collecting the time-independent terms we obtain the following ordinary differential equation for the function $\varphi_2^{(0)}(y)$:

$$
\begin{aligned}
&2S\left(U_y\left(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)}\right)+U\left(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)}\right)\right)+2B\left(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)}\right)\\
&=ik\begin{pmatrix}\varphi_{1y}\varphi_{1yy}^{*}-\varphi_{1y}^{*}\varphi_{1yy}\\ +\varphi_{1}\varphi_{1yyy}^{*}-\varphi_{1}^{*}\varphi_{1yyy}\end{pmatrix}-\frac{S}{2}\begin{pmatrix}k^2\left(\varphi_1\varphi_{1y}^{*}+\varphi_1^{*}\varphi_{1y}\right)\\ +2\left(\varphi_{1y}^{*}\varphi_{1yy}+\varphi_{1yy}^{*}\varphi_{1y}\right)\end{pmatrix}.
\end{aligned}
\tag{3.14}
$$

The function $\varphi_2^{(0)}(y)$ satisfies the following boundary conditions:

$$
\varphi_2^{(0)}(\pm\infty)=0. \tag{3.15}
$$

Substituting (3.13) into (3.12) and collecting the terms containing the first harmonic we obtain the equation

$$
\begin{aligned}
&\left(U-c-SU\frac{i}{k}-\frac{iB}{k}\right)\varphi_{2yy}^{(1)}+\left(2\frac{U}{R}-SU_y\frac{i}{k}\right)\varphi_{2y}^{(1)}+\left(k^2c-k^2U-U_{yy}+\frac{ikSU}{2}+ikB\right)\varphi_2^{(1)}\\
&=-\frac{i}{k}\left(c_g-U\right)\varphi_{1yy}+2\frac{iU}{kR}\varphi_{1y}+\left(2ikc-3ikU-\frac{i}{k}U_{yy}+ikc_g-US-2B\right)\varphi_1
\end{aligned}
\tag{3.16}
$$

with the boundary conditions

$$
\varphi_2^{(1)}(\pm\infty)=0. \tag{3.17}
$$

Finally, using (3.13) and (3.12) for the terms that contain the second harmonic, we obtain

$$
\begin{aligned}
&8ik^3c\varphi_2^{(2)}-2ikc\varphi_{2yy}^{(2)}-8ik^3U\varphi_2^{(2)}+2ikU\varphi_{2yy}^{(2)}-2ikU_{yy}\varphi_2^{(2)}\\
&+S\begin{pmatrix}2U_y\varphi_{2y}^{(2)}-4k^2U\varphi_2^{(2)}\\ +2U\varphi_{2yy}^{(2)}\end{pmatrix}+\frac{4ik}{R}U\varphi_{2y}^{(2)}+B\left(\varphi_{2yy}^{(2)}-4k^2\varphi_2^{(2)}\right)\\
&=ik\left(\varphi_1\varphi_{1yyy}-\varphi_{1y}\varphi_{1yy}\right)-S\left(2\varphi_{1y}\varphi_{1yy}-3k^2\varphi_1\varphi_{1y}\right)-\frac{2ik}{R}\varphi_{1y}^2.
\end{aligned}
\tag{3.18}
$$

The boundary conditions are

$$
\varphi_2^{(2)}(\pm\infty)=0. \tag{3.19}
$$

Comparing (3.6) and (3.16) we see that the left-hand sides of both equations are the same. Thus, (3.21) has a solution if and only if the right-hand side of (3.16) is orthogonal to all eigenfucntions of the corresponding adjoint problem (see [69]). The adjoint operator L^{a} and adjoint eigenfunction φ_1^{a} are defined as follows:

$$\int_{-\infty}^{+\infty} \varphi_1^a \cdot L\varphi_1 dy = \int_{-\infty}^{+\infty} \varphi_1 \cdot L^a \varphi_1^a dy. \tag{3.20}$$

The adjoint problem is

$$L^{\mathrm{a}} \varphi_1^{\mathrm{a}} = 0, \tag{3.21}$$

$$\varphi_1^{\mathrm{a}}(\pm\infty) = 0. \tag{3.22}$$

Integrating the left-hand side of (3.20) by parts and using boundary conditions (3.7), (3.22) we obtain

$$\begin{aligned} L^{\mathrm{a}}\varphi_1^{\mathrm{a}} \equiv \varphi_{1yy}^{\mathrm{a}} \left(U - c - SU\frac{i}{k} - B\frac{i}{k} \right) + \varphi_{1y}^{\mathrm{a}} \left(2U_y - SU_y\frac{i}{k} - 2\frac{U}{R} \right) \\ + \varphi_1^{\mathrm{a}} \left(k^2 c - k^2 U + \frac{ik}{2} SU - 2\frac{U_y}{R} + Bik \right). \end{aligned} \tag{3.23}$$

Solvability condition for (3.16) has the form

$$\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \left((c_g - U)\varphi_{1yy} - 2\frac{U}{R}\varphi_{1y} + \begin{pmatrix} -2k^2 c + 3k^2 U + U_{yy} \\ -k^2 c_g + ikUS + 2Bik \end{pmatrix} \varphi_1 \right) dy = 0. \tag{3.24}$$

Hence, the group velocity c_g can be found from (3.24).

Using (3.1), (3.2), (3.11) and collecting the terms that contain ε^3 we obtain:

$$\begin{aligned} L_1\psi_3 = c_g\left(\psi_{2xx\xi} + \psi_{2yy\xi}\right) - \psi_{1xx\tau} - 2\psi_{2x\xi\ t} + 2c_g\psi_{1x\xi\xi} - \psi_{1\xi\xi\ t} - \psi_{1yy\tau} \\ - 3U\psi_{2xx\xi} - 3U\psi_{1x\xi\xi} - \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \psi_{2y}\psi_{1xxx} - \psi_{2y}\psi_{1yyx} \\ - \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi\ yy} - U\psi_{2\xi\ yy} + \psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} \\ + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} + \psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}U_{yy} \\ - \frac{S}{2}\begin{pmatrix} \psi_{1xx}\psi_{2y} + 1.5\psi_{1xx}\psi_{1x}^2/U + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2U\psi_{2x\xi} \\ + U\psi_{1\xi\xi} + \psi_{1yy}\psi_{2y} + \psi_{2yy}\psi_{1y} - U\psi_{1xx} - 2U_y\psi_{1y} - 2U\psi_{1yy} \\ + \psi_{1yy}\psi_{2y} + \psi_{1y}\psi_{2yy} + 2\psi_{1x}\psi_{2xy} + 2\psi_{1x}\psi_{1\xi\ y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy} \end{pmatrix} \\ - \frac{2}{R}\left(U\psi_{2\xi\ y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi\ y} + \psi_{2y}\psi_{1xy}\right) - B\left(2\psi_{2x\xi} + \psi_{1\xi\xi}\right). \end{aligned} \tag{3.25}$$

The evolution equation for the amplitude function $A(\xi,\tau)$ is determined from the solvability condition at the third order. Multiplying the right-hand side of (3.25) by φ_1^{a}, using (3.13) and the solutions of the boundary value problems (3.14)-(3.19) we obtain the complex Ginzburg-Landau equation for the amplitude $A(\xi,\tau)$ of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A, \tag{3.26}$$

where

$$\sigma = \frac{\sigma_1}{\eta}, \ \delta = \frac{\delta_1}{\eta}, \ \mu = \frac{\mu_1}{\eta} \tag{3.27}$$

and the complex coefficients $\sigma_1, \delta_1, \mu_1$ and η are given by

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^{a} \left(\varphi_{1yy} - k^2 \varphi_1 \right) dy, \tag{3.28}$$

$$\sigma_1 = \frac{S}{2} \int_{-\infty}^{+\infty} \varphi_1^{a} \left(-k^2 U \varphi_1 + 2U_y \varphi_{1y} + 2U \varphi_{1yy} \right) dy, \tag{3.29}$$

$$\mu_1 = -\int_{-\infty}^{+\infty} \varphi_1^{a} \left(\begin{array}{l} 6ik^3 \varphi_2^{(2)} \varphi_{1y}^{*} - 2ik \varphi_{1y}^{*} \varphi_{2yy}^{(2)} + 3ik^3 \varphi_1^{*} \varphi_{2y}^{(2)} \\ + ik^3 \varphi_1 \left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)} \right) - ik \varphi_{1yy} \left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)} \right) \\ + ik \varphi_{2y}^{(2)} \varphi_{1yy}^{*} - ik \varphi_1^{*} \varphi_{2yyy}^{(2)} \\ + ik \varphi_1 \left(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)} \right) + 2ik \varphi_{1yyy}^{*} \varphi_2^{(2)} \\ -\dfrac{S}{2} \left(\begin{array}{l} -k^2 \varphi_1 \left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)} \right) + 3k^2 \varphi_1^{*} \varphi_{2y}^{(2)} - \dfrac{3k^4}{2u_0} \varphi_1^2 \varphi_1^{*} \\ + 2\varphi_{1yy} \left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)} \right) + 2\varphi_{1yy}^{*} \varphi_{2y}^{(2)} \\ + 2\varphi_{1y} \left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)} \right) + 2\varphi_{2yy}^{(2)} \varphi_{1y}^{*} \end{array} \right) \\ -2\dfrac{ik}{R} \left(\varphi_{2y}^{(2)} \varphi_{1y}^{*} + \varphi_{2y}^{(0)} \varphi_{1y} \right) \end{array} \right) dy, \tag{3.30}$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^{a} \left(\begin{array}{l} \left(c_g - U \right) \varphi_{2yy}^{(1)} - 2\dfrac{U}{R} \varphi_{2y}^{(1)} \\ + \varphi_2^{(1)} \left(-k^2 c_g - 2k^2 c + 3k^2 U + U_{yy} - ikSU - 2ikB \right) \\ + \varphi_1 \left(2ikc_g + ikc - 3ikU - U\dfrac{S}{2} - B \right) \end{array} \right) dy. \tag{3.31}$$

The coefficients of the Ginzburg-Landau equation (3.26) can be computed using formulas (3.27) - (3.31). Note that in order to perform calculations it is necessary to solve the linear stability problem (3.6) - (3.7), the corresponding adjoint problem (3.21) - (3.22), three boundary value problems (3.14) - (3.19) and numerically evaluate integrals in (3.27) - (3.31).

4. SPATIAL STABILITY OF SLIGHTLY CURVED SHALLOW MIXING LAYERS

4.1 Linear Case

There are two basic approaches for the analysis of linear stability of a base flow in fluid mechanics: (a) temporal stability analysis and (b) spatial stability analysis [11]. In both cases the analysis is performed using the method of normal modes: perturbations are assumed to be proportional to $\exp(i(\alpha x - \beta t))$, where both parameters α and β may be complex:

$$\alpha = \alpha_{\mathrm{r}} + i\alpha_{\mathrm{i}},\ \beta = \beta_{\mathrm{r}} + i\beta_{\mathrm{i}}.$$

In case (a) the wave number $\alpha = \alpha_{\mathrm{r}}$ is real while β is complex. For the case of spatial stability analysis $\beta = \beta_{\mathrm{r}}$ is real and the wave number α is complex: $\alpha = \alpha_{\mathrm{r}} + i\alpha_{\mathrm{i}}$. From a computational point of view temporal stability analysis is simpler since the corresponding eigenvalue problem is linear with respect to eigenvalue β. On the other hand, spatial eigenvalue problem is nonlinear in α. However, spatial growth rates are usually evaluated experimentally so that spatial stability characteristics should be calculated for a proper comparison with experimental data.

M. Gaster [32] suggested a transformation which can be used to approximate spatial growth rates if temporal growth rates are known. However, Gaster's transformation can be used only in the vicinity of the marginal stability curve.

A spatial stability problem for the case of slightly curved shallow mixing layers is solved in this chapter. Spatial growth rates are calculated for different values of the parameters of the problem. The effect of curvature on the stability of the base flow is analysed.

Shallow water equations under the rigid-lid assumption in the presence of a small curvature have the form (1.1) – (1.3) [12], [20], [21], [22] and [27]. Introducing the stream function (see Chapter 2.1) by the relations (2.1) we can rewrite (1.1) – (1.3) in the form (2.2). Consider a perturbed solution to (2.2) of the form

$$\psi(x, y, t) = \psi_0(y) + \varepsilon\psi_1(x, y, t) + \ldots \tag{4.1}$$

where $\psi_0(y)$ – the base flow solution,

ψ_1 – a small unsteady perturbation.

Substituting (4.1) into (2.2) and linearizing the resulting equation in the neighbourhood of the base flow we obtain (see Chapter 2.1)

$$L_1\psi_1 = 0, \tag{4.2}$$

where

$$\begin{aligned} L_1\psi &\equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{oy}\psi_{yyx} - \psi_{0yyy}\psi_x \\ &+ \frac{c_f}{2h}\left(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}\right) + \frac{2}{R}\psi_{0y}\psi_{xy} \end{aligned}.$$

Method of normal modes is used to solve (4.2), that is, the perturbation ψ_1 is represented in the form

$$\psi_1(x,y,t) = \varphi(y)\exp\left(i(\alpha x - \beta t)\right), \tag{4.3}$$

where $\varphi(y)$ – the amplitude of the normal perturbation.

Since spatial stability analysis is used, assumes that $\beta = \beta_r$ is the real frequency of the perturbation and $\alpha = \alpha_r + i\alpha_i$ is a complex number.

Substituting (4.3) into (4.2) and denoted $U = \psi_{0y}$ we obtain the following differential equation

$$\varphi_{yy}\left(\alpha U - \beta - iSU\right) - iSU_y\varphi_y + \frac{2U\alpha}{R}\varphi_y + \varphi\left(\alpha^2\beta - \alpha^3 U - \alpha U_{yy} + \frac{i\alpha^2 US}{2}\right) = 0 \tag{4.4}$$

with the boundary conditions

$$\varphi(\pm\infty) = 0, \tag{4.5}$$

where $S = \dfrac{c_f b}{h}$ – the bed-friction number;

b – a characteristic length scale (in this case width of the mixing layer).

Problem (4.4), (4.5) is an eigenvalue problem. Base flow $U(y)$ is said to be linearly stable if all $\alpha_i > 0$ and unstable if at least one $\alpha_i < 0$.

As it is mentioned above, the corresponding problem is linear with respect to β but nonlinear with respect to α. Hence, the following computational procedure is suggested for the solution of the problem. Assuming that both α and β are complex of the form $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$, for each fixed S, α_r and β_r we calculate α_i such that $\beta_i = 0$. This is achieved by solving linear generalized eigenvalue problem and selecting the new approximation to β_i using bisection method. Then we change α_r (for the fixed value of S) and repeat the calculation. The region of spatial instability is described by the relation $\alpha_i < 0$.

The base flow is selected in the form

$$U(y) = \frac{1}{2}(1 + \tanh y). \tag{4.6}$$

The first set of calculations is performed for the case without bottom friction ($S = 0$). The growth rates $-\alpha_i$ versus β_r are shown in Fig.4.1. It follows from Fig. 4.1 that curvature has a stabilizing influence on the flow (the growth rates decrease as the curvature increases).

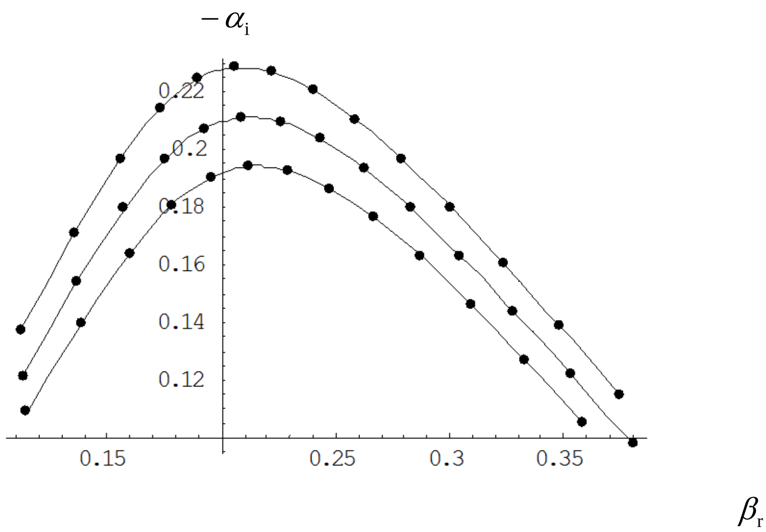

Fig. 4.1. Growth rates $-\alpha_i$ versus β_r for three values of $\frac{1}{R}$ = 0; 0.025 and 0.05 (from top to bottom).

The growth rates $-\alpha_i$ versus β_r are shown in Fig. 4.2. As can be seen from Fig. 4.2, the increase of the values of S also leads to more stable flow – the growth rates decrease as the parameter S grows.

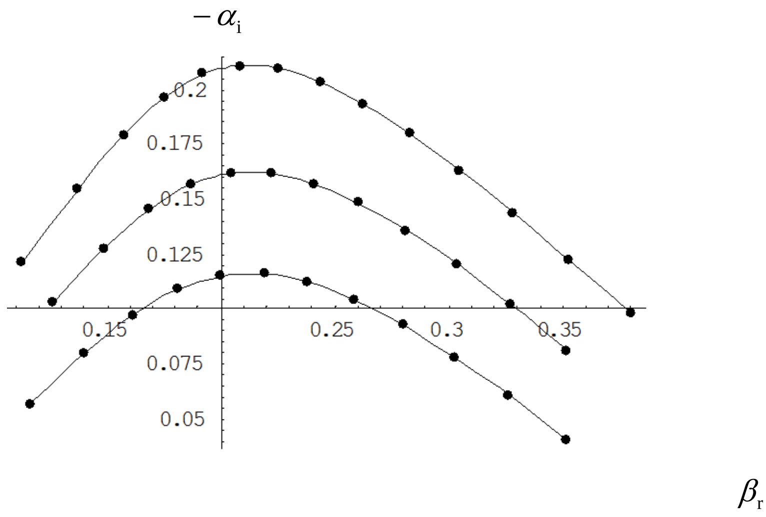

Fig. 4.2. Growth rates $-\alpha_i$ versus β_r for three values of S = 0; 0.05 and 0.1 (from top to bottom).

It is shown that both the bottom friction and flow curvature have a stabilizing influence on the flow.

Following M. Gaster [32] we denote by (T) and (Sp) the solutions to (4.4), (4.5) corresponding to temporal and spatial problems, respectively. It is shown in [32] that near the marginal stability curve:

$$\alpha_r(T) = \alpha_r(Sp),\ \beta_r(T) = \beta_r(Sp),\ \alpha_i(Sp) = -\frac{\beta_i(T)}{c(T)},$$

where $c(T) = \dfrac{\beta_r(T)}{\alpha_r(T)}$.

It follows from the Gaster's transformation that on the stability boundary either spatial or temporal stability analyses can be used since in this case $\alpha_i(Sp) = \beta_i(T) = 0$. If the objective of the analysis is to construct a marginal stability curve then it is recommended to use temporal stability analysis (which is a simpler method from a computational point of view than spatial stability analysis). However, the use of the Gaster's transformation away from the marginal stability curve can result in relatively large errors. We have computed temporal and spatial growth rates for the case S = 0.05, B = 0 and $1/R$ = 0. The relative percentage errors δ in using Gaster's transformation are shown in the Table 4.1.

Table 4.1: Relative Errors in Using Gaster's Transformation.

α_r	$\delta(\%)$
0.1	11.6
0.2	15.4
0.3	16.3
0.4	15.0
0.5	12.7

It is seen from Table 4.1 that errors in using Gaster's transformation for the calculation of growth rates away from the marginal stability curve can be quite large.

4.2 Weakly Nonlinear Case

In this Chapter we describe the second approach which can be used in order to derive an amplitude evolution equation under the assumption that the base flow is not parallel but slightly changes downstream.

Consider the system of shallow water equations of the form (1.1) – (1.3). Let $\psi(x,y,t)$ be the stream function of the flow. The velocity components can be written in the form (2.1). Using (2.1) and eliminating the pressure from (1.1.) - (1.3) the system of shallow water equations reduces to the following equation for the stream function

$$\begin{aligned}&(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy}+\frac{c_\mathrm{f}}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}\\&+\frac{c_\mathrm{f}}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx})+B\Delta\psi=0.\end{aligned} \tag{4.7}$$

Assume that λ is the wave length of a perturbation and l is the length scale of the longitudinal variation of the base flow. In shallow mixing layers (see [64], [65]) the following condition is usually satisfied: $\lambda << l$. Thus, a small parameter ε can be defined as follows: $\varepsilon=\lambda/l$. Following [36] we introduce a slow longitudinal coordinate X by the relation $X=\varepsilon x$. The base flow velocity components are $U(y,X)$ and $\varepsilon V(y,X)$, respectively.

The stream function $\psi(x,y,t)$ is represented as the sum of the basic part $\psi_0(y,X)$ and fluctuating part $\psi'(x,y,t)$:

$$\psi(x,y,t)=\psi_0(y,X)+\psi'(x,y,t)\ , \tag{4.8}$$

where $X = \varepsilon x$ – a slowly varying coordinate;

ε – the small dimensionless parameter that characterizes the non-parallelism of the base flow;

$\psi_0(y, X)$ – the stream function of the base flow;

$\psi'(x, y, t)$ – a perturbation.

In addition, $U(y, X) = \dfrac{\partial \psi_0}{\partial y}$ and $V(y, X) = -\dfrac{\partial \psi_0}{\partial X}$.

Using (4.8) the derivatives of ψ with respect to *x*, *y* and *t* we get in the form (linearizing equation the effect of the perturbation ψ' and ε is small, therefore the quadratic or higher terms of the ψ' and ε may be ignored):

$$\frac{\partial \psi(x,y,t)}{\partial x} = \frac{\partial \psi_0}{\partial X}\varepsilon + \frac{\partial \psi'}{\partial x} = -V\varepsilon + \frac{\partial \psi'}{\partial x} = \psi_x$$

$$\frac{\partial \psi(x,y,t)}{\partial y} = \frac{\partial \psi_0}{\partial y} + \frac{\partial \psi'}{\partial y} = U + \frac{\partial \psi'}{\partial y} = \psi_y$$

$$\psi_{yy} \to \frac{\partial U}{\partial y} + \frac{\partial^2 \psi'}{\partial y^2} \qquad \psi_{xxy} \to \varepsilon^2 \frac{\partial^2 U}{\partial X^2} + \frac{\partial^3 \psi'}{\partial x^2 \partial y} \to \frac{\partial^3 \psi'}{\partial x^2 \partial y}$$

$$\psi_{yx} \to \varepsilon \frac{\partial U}{\partial X} + \frac{\partial^2 \psi'}{\partial x \partial y} \qquad \psi_{xyy} \to \varepsilon \frac{\partial^2 U}{\partial X \partial y} + \frac{\partial^3 \psi'}{\partial x \partial y^2}$$

$$\psi_{xx} \to -\varepsilon^2 \frac{\partial V}{\partial X} + \frac{\partial^2 \psi'}{\partial x^2} \to \frac{\partial^2 \psi'}{\partial x^2} \qquad \psi_{yyy} \to \frac{\partial^2 U}{\partial y^2} + \frac{\partial^3 \psi'}{\partial y^3}$$

$$\psi_{xxx} \to -\varepsilon^3 \frac{\partial^2 V}{\partial X^2} + \frac{\partial^3 \psi'}{\partial x^3} \to \frac{\partial^3 \psi'}{\partial x^3} \qquad \Delta \psi = \psi_{xx} + \psi_{yy} \to \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial U}{\partial y} + \frac{\partial^2 \psi'}{\partial y^2}$$

$$(\Delta \psi)_t = \left(\psi_{xx} + \psi_{yy}\right)_t \to \left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial U}{\partial y} + \frac{\partial^2 \psi'}{\partial y^2}\right)_t = \left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2}\right)_t$$

$$(\Delta \psi)_x = \left(\psi_{xx} + \psi_{yy}\right)_x = \psi_{xxx} + \psi_{yyx} \to \frac{\partial^3 \psi'}{\partial x^3} + \varepsilon \frac{\partial^2 U}{\partial X \partial y} + \frac{\partial^3 \psi'}{\partial x \partial y^2}$$

$$(\Delta\psi)_y = \left(\psi_{xx} + \psi_{yy}\right)_y = \psi_{xxy} + \psi_{yyy} \to \frac{\partial^3\psi'}{\partial x^2 \partial y} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^3\psi'}{\partial y^3}$$

$$\psi_x^2 \to \left(-\varepsilon V + \frac{\partial\psi'}{\partial x}\right)^2 = \varepsilon^2 V^2 - 2\varepsilon V\frac{\partial\psi'}{\partial x} + \left(\frac{\partial\psi'}{\partial x}\right)^2 \to -2\varepsilon V\frac{\partial\psi'}{\partial x}$$

$$\psi_y^2 \to \left(U + \frac{\partial\psi'}{\partial y}\right)^2 = U^2 + 2U\frac{\partial\psi'}{\partial y} + \left(\frac{\partial\psi'}{\partial y}\right)^2 \to U^2 + 2U\frac{\partial\psi'}{\partial y}$$

$$\psi_x^2 + \psi_y^2 \to -2\varepsilon V\frac{\partial\psi'}{\partial x} + U^2 + 2U\frac{\partial\psi'}{\partial y}$$

$$\psi_y(\Delta\psi)_x \to \left(U + \frac{\partial\psi'}{\partial y}\right)\left(\frac{\partial^3\psi'}{\partial x^3} + \varepsilon\frac{\partial^2 U}{\partial X\partial y} + \frac{\partial^3\psi'}{\partial x\partial y^2}\right) \to$$
$$\frac{\partial^3\psi'}{\partial x^3}U + \varepsilon\frac{\partial^2 U}{\partial X\partial y}U + \frac{\partial^3\psi'}{\partial x\partial y^2}U + \varepsilon\frac{\partial^2 U}{\partial X\partial y}\frac{\partial\psi'}{\partial y}$$

$$\psi_x(\Delta\psi)_y \to \left(-\varepsilon V + \frac{\partial\psi'}{\partial x}\right)\left(\frac{\partial^3\psi'}{\partial x^2\partial y} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^3\psi'}{\partial y^3}\right) \to$$
$$-\varepsilon V\frac{\partial^3\psi'}{\partial x^2\partial y} - \varepsilon V\frac{\partial^2 U}{\partial y^2} - \varepsilon V\frac{\partial^3\psi'}{\partial y^3} + \frac{\partial^2 U}{\partial y^2}\frac{\partial\psi'}{\partial x}$$

$$\psi_y\psi_{xy} \to \left(U + \frac{\partial\psi'}{\partial y}\right)\left(\varepsilon\frac{\partial U}{\partial X} + \frac{\partial^2\psi'}{\partial x\partial y}\right) \to \varepsilon U\frac{\partial U}{\partial X} + U\frac{\partial^2\psi'}{\partial x\partial y} + \varepsilon\frac{\partial\psi'}{\partial y}\frac{\partial U}{\partial X}$$

$$(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) \to \left(U^2 + 2U\frac{\partial\psi'}{\partial y}\right)\left(\frac{\partial U}{\partial y} + \frac{\partial^2\psi'}{\partial y^2}\right)$$
$$+2\left(-\varepsilon V + \frac{\partial\psi'}{\partial x}\right)\left(\varepsilon U\frac{\partial U}{\partial X} + U\frac{\partial^2\psi'}{\partial x\partial y} + \varepsilon\frac{\partial\psi'}{\partial y}\frac{\partial U}{\partial X}\right) + \left(-2\varepsilon V\frac{\partial\psi'}{\partial x}\right)\left(\frac{\partial^2\psi'}{\partial x^2}\right) \to$$
$$U^2\frac{\partial U}{\partial y} + 2U\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y} + U^2\frac{\partial^2\psi'}{\partial y^2} - 2\varepsilon VU\frac{\partial^2\psi'}{\partial x\partial y} + 2\varepsilon U\frac{\partial U}{\partial X}\frac{\partial\psi'}{\partial x}$$

Using function Maclaurin series $\sqrt{1+x} = 1 + \frac{x}{2} + \ldots$:

$$\sqrt{\psi_x^2 + \psi_y^2} \to U + \frac{\partial\psi'}{\partial y} - \left(\frac{1}{U} - \frac{1}{U^2}\frac{\partial\psi'}{\partial y}\right)\varepsilon V \cdot \frac{\partial\psi'}{\partial x}$$

$$\Delta\psi\sqrt{\psi_x^2+\psi_y^2}\to$$

$$U\frac{\partial^2\psi'}{\partial x^2}+U\frac{\partial U}{\partial y}+U\frac{\partial^2\psi'}{\partial y^2}-\frac{\varepsilon V}{U}\cdot\frac{\partial\psi'}{\partial x}\cdot\frac{\partial U}{\partial y}+\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}$$

$$\frac{1}{\sqrt{\psi_x^2+\psi_y^2}}\to\frac{1}{U}+\frac{\varepsilon V}{U^3}\cdot\frac{\partial\psi'}{\partial x}-\frac{1}{U^2}\cdot\frac{\partial\psi'}{\partial y}$$

$$\frac{\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}}{\sqrt{\psi_x^2+\psi_y^2}}\to$$

$$\to U\frac{\partial U}{\partial y}+2\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}+U\frac{\partial^2\psi'}{\partial y^2}-2\varepsilon V\frac{\partial^2\psi'}{\partial x\partial y}+2\varepsilon\frac{\partial U}{\partial X}\frac{\partial\psi'}{\partial x}+$$
$$+\frac{\varepsilon V}{U}\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial x}-\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}$$

Substituting all this expressions into (4.7) we obtain:

$$\begin{aligned}
&\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)_t+\left(\frac{\partial^3\psi'}{\partial x^3}U+\varepsilon\frac{\partial^2 U}{\partial X\partial y}U+\frac{\partial^3\psi'}{\partial x\partial y^2}U+\varepsilon\frac{\partial^2 U}{\partial X\partial y}\frac{\partial\psi'}{\partial y}\right)\\
&-\left(-\varepsilon V\frac{\partial^3\psi'}{\partial x^2\partial y}-\varepsilon V\frac{\partial^2 U}{\partial y^2}-\varepsilon V\frac{\partial^3\psi'}{\partial y^3}+\frac{\partial^2 U}{\partial y^2}\frac{\partial\psi'}{\partial x}\right)\\
&+\frac{2}{R}\left(\varepsilon U\frac{\partial U}{\partial X}+U\frac{\partial^2\psi'}{\partial x\partial y}+\varepsilon\frac{\partial\psi'}{\partial y}\frac{\partial U}{\partial X}\right)\\
&+\frac{c_\mathrm{f}}{2h}\left(U\frac{\partial^2\psi'}{\partial x^2}+U\frac{\partial U}{\partial y}+U\frac{\partial^2\psi'}{\partial y^2}-\frac{\varepsilon V}{U}\frac{\partial\psi'}{\partial x}\frac{\partial U}{\partial y}+\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}\right)\\
&+\frac{c_\mathrm{f}}{2h}\left(\begin{aligned}&U\frac{\partial U}{\partial y}+2\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}+U\frac{\partial^2\psi'}{\partial y^2}-2\varepsilon V\frac{\partial^2\psi'}{\partial x\partial y}+\\&+2\varepsilon\frac{\partial U}{\partial X}\frac{\partial\psi'}{\partial x}+\frac{\varepsilon V}{U}\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial x}-\frac{\partial U}{\partial y}\frac{\partial\psi'}{\partial y}\end{aligned}\right)+B\left(\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2}\right)=0.
\end{aligned}\tag{4.9}$$

Simplifying the expression (4.9), grouping the terms and denoting the primes, we get:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+U\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\frac{\partial\psi}{\partial x}\frac{\partial^2 U}{\partial y^2}$$
$$+\frac{c_{\mathrm{f}}}{2h}\left[U\left(\frac{\partial^2\psi}{\partial x^2}+2\frac{\partial^2\psi}{\partial y^2}\right)+2\frac{\partial U}{\partial y}\frac{\partial\psi}{\partial y}\right]+B\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$
$$+\varepsilon\left(\begin{array}{l}\dfrac{\partial^2 U}{\partial y\partial X}\dfrac{\partial\psi}{\partial y}+V\dfrac{\partial}{\partial y}\left(\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}\right)\\ +\dfrac{c_{\mathrm{f}}}{2h}\left(2\dfrac{\partial U}{\partial X}\dfrac{\partial\psi}{\partial x}-2V\dfrac{\partial^2\psi}{\partial y\partial x}+\dfrac{U_y}{U}V\dfrac{\partial\psi}{\partial x}\right)\\ +\dfrac{2}{R}\dfrac{\partial U}{\partial X}\dfrac{\partial\psi}{\partial y}\end{array}\right)=0 \tag{4.10}$$

Using the *WKBJ* approximation (see [36]) we represent the stream function in the form

$$\psi(x,y,t)=\varphi(y,X)\exp\left(i\left(\frac{\theta(X)}{\varepsilon}-\omega\,t\right)\right), \tag{4.11}$$

where $\varphi(y,X)$ – a slow-varying amplitude function;

$\dfrac{\theta(X)}{\varepsilon}$ – a fast-varying phase function.

The amplitude function $\varphi(y,X)$ is expanded in a power series of the form

$$\varphi(y,X)=\varphi_1(y,X)+\varepsilon\varphi_2(y,X)+... \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10) we obtain the following equation at the leading order:

$$L\varphi_1=0, \tag{4.13}$$

where

$$L\varphi_1=\varphi_1''-k^2\varphi_1-\frac{U''}{U-\dfrac{\omega}{k}}\varphi_1+\frac{2}{R}U\varphi_1'+B\frac{ik}{U-\dfrac{\omega}{k}}\varphi_1$$
$$-\frac{ic_{\mathrm{f}}}{2h(kU-\omega)}\left(-Uk^2\varphi_1+2U\varphi_1''+2U'\varphi_1'\right). \tag{4.14}$$

Here primes denote the derivatives with respect to y and $\theta_x=k.$ Using equation (4.12) with zero boundary conditions at $\pm\infty$ we obtain linear stability problem where X

appears as the parameter. The corresponding eigenfunction of the linear stability problem, $\varphi_1(y,X)$, is represented in the form

$$\varphi_1(y,X) = A(X)\Phi(y,X), \tag{4.15}$$

where $A(X)$ – slowly varying amplitude;

$\Phi(y,X)$ – a normalized eigenfunction.

At the next order the following equation is obtained:

$$L\varphi_2 = F, \tag{4.16}$$

where

$$F = \frac{i}{kU-\omega}\frac{dA}{dX}\begin{pmatrix} 2\omega k\Phi - 3Uk^2\Phi + U\Phi'' - \Phi U'' \\ -\dfrac{2U}{R\Phi'} + c_f\dfrac{iUK\Phi}{h} + 2ikB\Phi \end{pmatrix}$$

$$+\frac{i}{kU-\omega}A\left\{\begin{array}{l} 2\omega k\dfrac{\partial\Phi}{\partial X} + \omega\Phi\dfrac{dk}{dX} - 3Uk^2\dfrac{\partial\Phi}{\partial x} \\ -3Uk\dfrac{dk}{dX}\Phi + U\dfrac{\partial\Phi''}{\partial X} - U''\dfrac{\partial\Phi}{\partial X} \\ +\dfrac{\partial U'}{\partial X}\Phi' - 2\dfrac{U}{R}\dfrac{\partial\Phi'}{\partial X} + V\left(\Phi''' - k^2\Phi\right) \\ +\dfrac{c_\mathrm{f}}{2h}\left[2iUk\dfrac{\partial\Phi}{\partial X} + 2ik\dfrac{\partial U}{\partial X}\Phi + i\dfrac{dk}{dX}U\Phi - 2ikV\Phi'\right] \\ +B(2ik\dfrac{\partial\Phi}{\partial X} + i\Phi\dfrac{dk}{dX}) \end{array}\right\}$$

Equation (4.16) has a solution if and only if the right-hand side F is orthogonal to all eigenfunctions $\tilde{\Phi}$ of the corresponding adjoint problem:

$$\int_{-\infty}^{+\infty} F\tilde{\Phi}dy = 0. \tag{4.17}$$

Using (4.16) and (4.17) we obtain the following equation for unknown amplitude $A(X)$:

$$M(X)\frac{dA}{dX} + N(X)A = 0, \tag{4.18}$$

where

$$M(X)=i\int_{-\infty}^{+\infty}\frac{2\omega k\Phi-3Uk^2\Phi+U\Phi''-\Phi U''-2\frac{U}{R}\Phi'+\frac{c_{\mathrm{f}}}{h}iUk\Phi+2ikB\Phi}{kU-\omega}\tilde{\Phi}dy, \quad (4.19)$$

$$N(X)=i\int_{-\infty}^{+\infty}\frac{\mathrm{D}\tilde{\Phi}}{kU-\omega}dy, \tag{4.20}$$

and

$$\begin{aligned}\mathrm{D}&=2\omega k\frac{\partial\Phi}{\partial x}+\omega\Phi\frac{dk}{dX}-3Uk^2\frac{\partial\Phi}{\partial X}-3Uk\frac{dk}{dX}\Phi+U\frac{\partial\Phi''}{\partial X}\\&-U''\frac{\partial\Phi}{\partial X}+\frac{\partial U'}{\partial X}\Phi'-2\frac{U}{R}\frac{\partial\Phi'}{\partial X}+V\left(\Phi'''-k^2\Phi\right)\\&\frac{c_{\mathrm{f}}}{2h}\left(2iUk\frac{\partial\Phi}{\partial x}+2ik\frac{\partial U}{\partial X}\Phi+i\frac{dk}{dX}U\Phi-2ikV\Phi'\right)+B\left(2ik\frac{\partial\Phi}{\partial X}+i\frac{dk}{dX}\Phi\right)\end{aligned} \tag{4.21}$$

As a result, the fluctuating part of the stream function has the form

$$\psi(x,y,t)\sim A(X)\Phi(y,X)\times\exp\left(i\left(\frac{1}{\varepsilon}\int_0^X k(X)dX-\omega t\right)\right) \tag{4.22}$$

Formula (4.22) takes into account (in asymptotic form) slow longitudinal variation of the base flow. It is shown in [8] that in a similar asymptotic formula the growth rate and phase speed of a perturbation depend not only on the choice of flow quantities but also on the location of the point (x,y), where these quantities are calculated. This fact has to be taken into account for a proper comparison of (4.22) with experimental data.

A few important conclusions can be drawn from the asymptotic analysis (see [8]):

1. Each multiplier on the right-hand side of (4.22) contains information related to both amplitude and phase of the perturbation.
2. The selection of the perturbed quantities plays an important role in the calculation of the growth rate and phase speed of the perturbation.
3. The growth rate and the phase speed of the perturbation depend not only on the perturbed quantity (velocity component or pressure), but also on the location of the downstream station where the quantities are calculated.

Hence, a meaningful comparison of the weakly nonlinear model (4.18) can be made only if a particular quantity of interest Q is selected (for example, longitudinal velocity component or pressure). In this case (see [8]) a local wave number k_{L} can be defined by the formula

$$k_{\mathrm{L}}(x, y) = -i \frac{\partial}{\partial x} \ln Q(x, y) \tag{4.23}$$

where $k_{\mathrm{L}} = k_{\mathrm{Lr}} + i k_{\mathrm{Li}}$.

The values of k_{Lr} and k_{Li} are interpreted as the local phase speed and local spatial growth rate, respectively. Thus, in order to compare weakly nonlinear model (4.18) with experimental data the following steps should be performed:

- select a flow quantity Q;
- measure the quantity Q at some point (x, y);
- compute the right-hand side of (4.23) at the same point (x, y).

In summary, weakly nonlinear model (4.18) can be validated if detailed experimental data or numerical results of the solution of nonlinear shallow water equations are available.

5. STABILITY OF SHALLOW MIXING LAYERS WITH VARIABLE FRICTION

5.1 Linear Case

Linear and weakly nonlinear stability problem for the case where the friction coefficient is varied in the transverse direction by an arbitrary differentiable shape function depending on transverse coordinate is analysed in this chapter.

Consider the system of shallow water equations under the rigid-lid assumption (1.1) - (1.3) [18], [19], [23] and [29]. The dependence of the friction coefficient $c_{\mathrm{f}}(y)$ on the transverse coordinate y is assumed to be of the form

$$c_{\mathrm{f}}(y) = c_{\mathrm{f}_0}\gamma(y), \tag{5.1}$$

where $\gamma(y)$ - arbitrary differentiable "shape" function.

Introducing the stream function ψ by the relations (2.1) and eliminating the pressure from (1.1)-(1.3) we obtain (see Chapter 2.1):

$$\begin{aligned}&(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{c_{\mathrm{f}}(y)}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}\\ &+\frac{c_{\mathrm{f}}(y)}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}) + \frac{c_{\mathrm{f}_y}(y)}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2} = 0,\end{aligned} \tag{5.2}$$

where $c_{\mathrm{f}_y}(y) = c_{\mathrm{f}_0}\gamma'(y)$ - the derivative of $c_{\mathrm{f}}(y)$ with respect to y.

Consider a perturbed solution to (5.2) of the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + ..., \tag{5.3}$$

where $U(y) = \psi_0(y)$ - the base flow solution.

Substituting (5.3) into (5.2) and linearizing the resulting equation in the neighbourhood of the base flow $U = U(y)$ we obtain

$$L_1\psi_1 = 0, \tag{5.4}$$

where

$$L_1\psi \equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{oy}\psi_{yyx} - \psi_{0yyy}\psi_x$$
$$+ \frac{c_{\mathrm{f}}}{2h}\left(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}\right) + \frac{c_{\mathrm{f}_y}}{h}\psi_{0y}\psi_y .$$

Using the method of normal modes we represent the function ψ_1 in the form

$$\psi_1(x, y, t) = \varphi(y)\exp\left(i(\alpha x - \beta t)\right), \tag{5.5}$$

where $\varphi(y)$ - the amplitude of the normal perturbation;

α - the complex wave number of the form $\alpha = \alpha_{\mathrm{r}} + i\alpha_{\mathrm{i}}$;

$\beta = \beta_{\mathrm{r}}$ - real (spatial stability analysis).

Substituting (5.5) into (5.4) and denoted $U = \psi_{0y}$ we obtain the following differential equation

$$\begin{aligned}&\varphi_{yy}\left(\alpha U - \beta - i\gamma SU\right) - i\gamma SU_y\varphi_y - i\gamma' SU\varphi_y \\ &+ \varphi\left(\alpha^2\beta - \alpha^3 U - \alpha U_{yy} + \frac{i\alpha^2 U\gamma S}{2}\right) = 0,\end{aligned} \tag{5.6}$$

where $S = \frac{c_{\mathrm{f}_0} b}{h}$ - the bed-friction number;

b - the half-width of the mixing layer.

The boundary conditions are

$$\varphi(\pm\infty) = 0. \tag{5.7}$$

The following profiles of the base flow velocity $U(y)$ and shape function $\gamma(y)$ are used to compute growth rates of unstable perturbations:

$$\gamma(y) = \frac{1}{2}(1 + \tanh y), \tag{5.8}$$

$$U(y) = \frac{1}{2}(1 - \tanh y). \tag{5.9}$$

The choice of the shape function $\gamma(y)$ in (5.9) is based on the following. First, with a stronger resistance force the base flow velocity becomes smaller so that (5.8) and (5.9) are consistent. Second, we would like to remove discontinuity in the friction force used in [66] and consider more realistic case of a continuous resistance which is changing with respect to the transverse coordinate.

Fig. 5.1 plots spatial growth rates for the unstable mode for three value of the parameter S: 0.05, 0.10 and 0.15 (from top to bottom). It is seen from the figure that with smaller S the growth rate is larger. This is understandable since the parameter S is proportional to the friction coefficient c_{f_y} c_{f_0}.

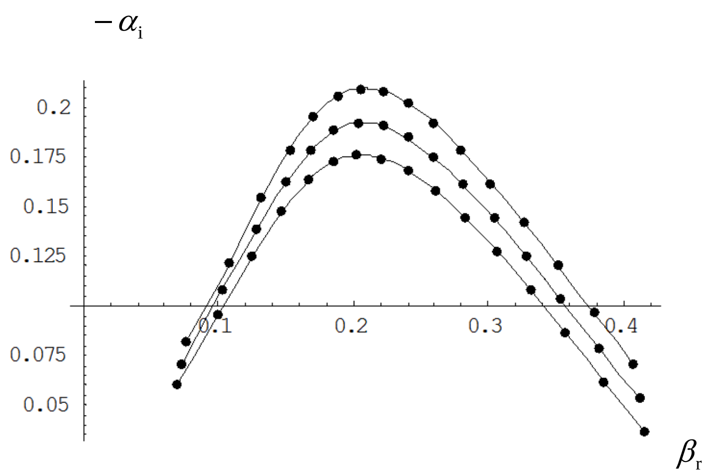

Fig. 5.1. Growth rates for the three values of S: 0.05, 0.10 and 0.15 (from top to bottom) for the shape function given by (5.12).

In order to see the effect of varying friction more clearly we plot in Fig. 5.2 growth rates for the most unstable mode for the same three values of S, namely, 0.05, 0.10 and 0.15 (from top to bottom) under the assumption that $\gamma(y)=1$ (that is, for the case of constant friction coefficient).

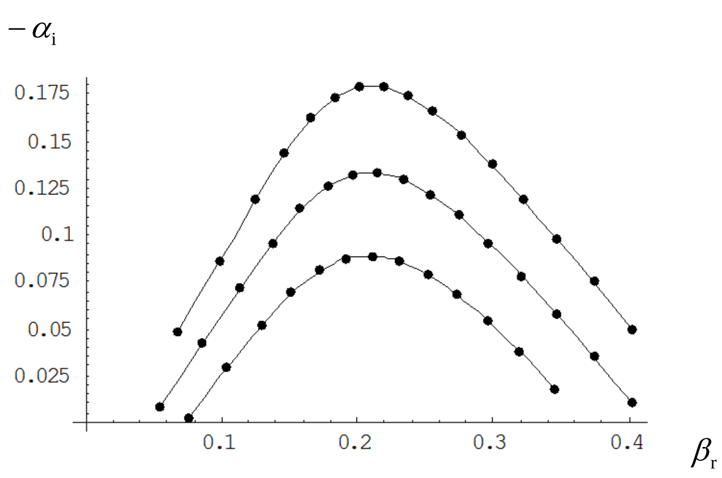

Fig. 5.2. Growth rates for the three values of S: 0.05, 0.10 and 0.15 (from top to bottom) for constant friction coefficient.

It is seen from Fig. 5.2 that the increase in S has a stabilizing influence on the flow (growth rates are getting smaller as S increases).

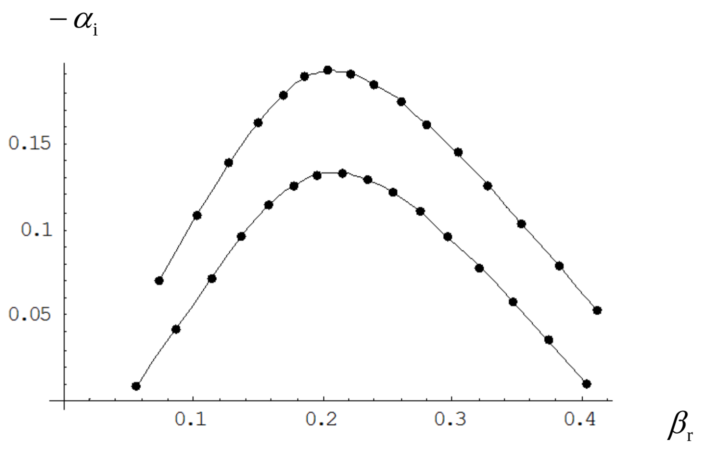

Fig. 5.3. Growth rates for the case $S = 0.1$
(variable friction – top curve, constant friction – bottom curve).

However, comparing Figs. 5.1 and 5.2 the overall growth rates for the case of non-uniform friction are larger than for the case of uniform friction. This fact is clearly seen from Fig. 5.3 where growth rates for $S = 0.1$ are plotted for the case of variable friction (top curve) and constant friction (bottom curve).

In the previous example, we considered the case of a symmetric profile, however, experimental data [66] showed that the base flow velocity profile is asymmetric with respect to the transverse coordinate. One example of such a flow is the flow in open channel with vegetation in floodplains [66]. The two-layer structure of the base flow is identified in [66]. A boundary-layer type of flow is observed in the outer layer (that is, in the main channel) and is characterized by relatively small velocity gradients. On the other hand, rather large velocity gradients are present in the inner layer due to the presence of vegetation in floodplains.

Two-parameter profiles of the base flow velocity $U(y)$ are used to compute growth rates of unstable perturbations:

$$U(y) = \begin{cases} 1 + r \tanh y, & y < 0 \\ 1 + \dfrac{r}{\delta} \tanh \delta y, & y > 0 \end{cases} \tag{5.10}$$

Here r is the velocity ratio, and δ is the "shape" parameter which reflects the two-layer structure described in [66]. Note also that the function $U(y)$ and its first and second derivatives are continuous at $y=0$.

Following [66] we assume that the drag force has the form

$$D=\begin{cases} 1/2\rho(C_D a+\dfrac{c_{\mathrm{f}}}{h})U_1^2, & y<0 \\ 1/2\rho\dfrac{c_{\mathrm{f}}}{h}U_2^2, & y>0 \end{cases} \tag{5.11}$$

where ρ - the density of the fluid;

C_D - the mean drag coefficient;

a - the average solid frontal area per unit volume in the plane perpendicular to the flow [66].

The drag differential between the layer with vegetation and the main channel is described by a dimensionless parameter

$$\gamma=\frac{C_D a}{C_D a+2c_{\mathrm{f}}/h} \; . \tag{5.12}$$

In addition, the total resistance can be measured by the generalized bed-friction number

$$S=\left(\frac{C_D a}{2}+\frac{c_{\mathrm{f}}}{h}\right)b\,, \tag{5.13}$$

where b - the width of the shear layer.

Using the linear stability problem (2.9) - (2.10) for $R=\infty$ we rewrite equation (2.9) in the form

$$(c-U)\varphi_{yy}+(U_{yy}+k^2(U-c))\varphi=\frac{SH(y)U}{ik}(\varphi_{yy}+\frac{U_y\varphi_y}{U}-\frac{k^2\varphi}{2}) \tag{5.14}$$

where

$$H(y)=\begin{cases} 1+\gamma, & y<0 \\ 1, & y=0 \\ 1-\gamma, & y>0 \end{cases} \tag{5.15}$$

This eigenvalue problem has complex eigenvalues of the form $c = c_r + ic_i$. Flow (5.11) is said to be linearly stable if $c_i < 0$, and unstable if $c_i > 0$. Problem is solved numerically by means of a collocation method based on Chebyshev polynomials (see Chapter 2.2). Software package *IMSL* is used to solve this problem. In order to avoid discontinuity at $y = 0$ the values of H are replaced by a hyperbolic tangent function of the form $\tanh \delta y$ with large δ values.

In order to compare the results obtained for asymmetric velocity profile (5.10) with the symmetric case we used the following symmetric velocity profile

$$U(y) = 1 - \frac{r}{2} + \frac{r}{2\delta} + \left(\frac{r}{2} + \frac{r}{2\delta}\right)\tanh y. \qquad (5.16)$$

Both profiles (5.10) and (5.16) have the same asymptotes as $y \to \pm\infty$. The graphs of the base flow velocity profiles (5.10) and (5.16) have shown in Figs. 5.4 and 5.5 for two values of δ.

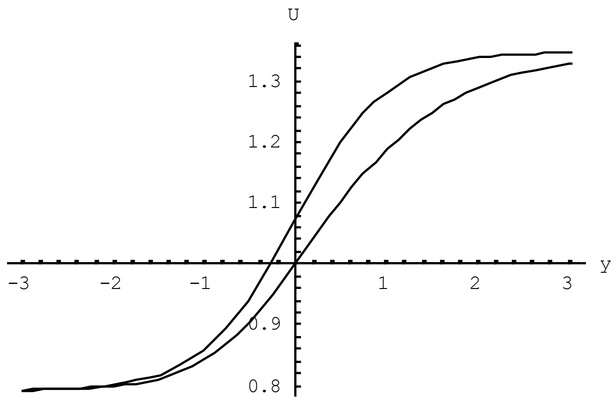

Fig. 5.4. Base flow velocity profiles calculated by means of (5.10) and (5.16) (top and bottom curves, respectively) for the case $\gamma = 0.8,\ \delta = 0.8$.

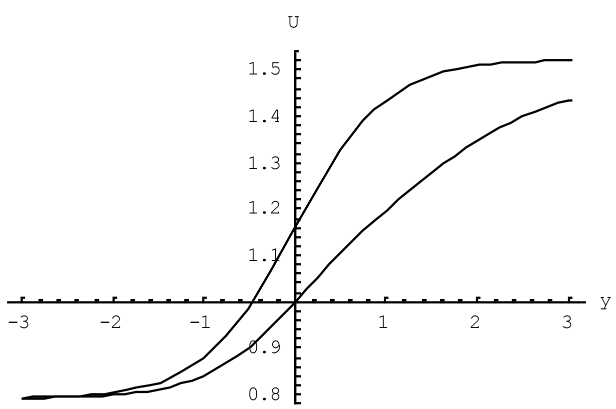

Fig. 5.5. Base flow velocity profiles calculated by means of (5.10) and (5.16) (top and bottom curves, respectively) for the case $\gamma = 0.8,\ \delta = 0.6$.

The role of the parameter δ is clearly seen from Figs. 5.4 and 5.5: for smaller values of δ the horizontal asymptote is reached at larger values of y if the base flow velocity profile is asymmetric with respect to the transverse coordinate y.

Stability curves in the (k,S) plane for different values of the parameters of problem (5.18), (5.14) are shown in Figs. 5.6 - 5.8. Marginal stability curves are shown for the symmetric case (base flow of the form (5.16), solid curve) and asymmetric case (base flow of the form (5.10), dashed curve).

Fig. 5.6 plots the marginal stability curves for the case $\gamma = 0.8,\ \delta = 0.8$. As can be seen from Fig. 5.6, asymmetry of the base flow velocity distribution results in more stable flow (the flow is stable above the curves in Figs. 5.6 - 5.8 and unstable below the curves).

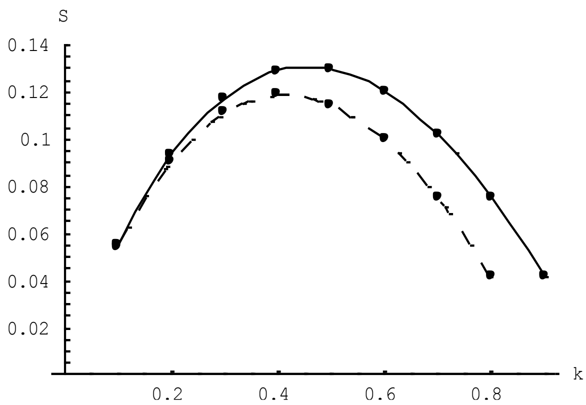

Fig. 5.6. Marginal stability curves for the case $\gamma = 0.8,\ \delta = 0.8$.

The stabilizing influence of asymmetry of the base flow is clearly seen also in Figs. 5.7 and 5.8. The asymmetric flow becomes more stable since the critical value of the parameter S becomes smaller. In addition, the range of unstable values of k also decreases.

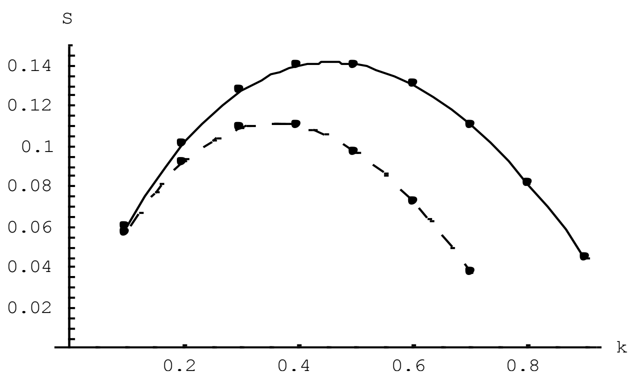

Fig. 5.7. Marginal stability curves for the case $\gamma = 0.8,\ \delta = 0.6$.

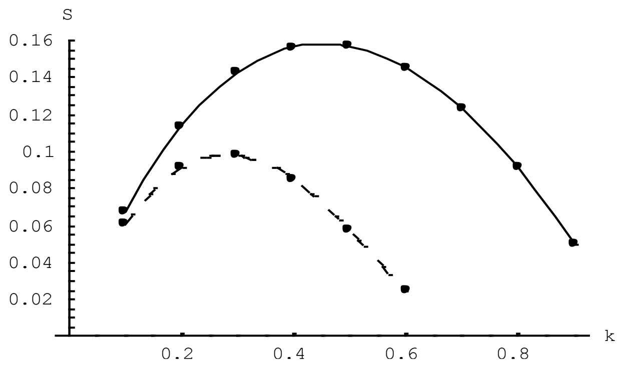

Fig. 5.8. Marginal stability curves for the case $\gamma = 0.8,\ \ \delta = 0.4$.

Numerical calculations showed stabilizing influence of asymmetry of the base flow profiles: both critical values of the stability parameter and the range of unstable wave numbers decrease as the asymmetry becomes more pronounced.

5.2 Weakly Nonlinear Case

Linear stability analysis of shallow mixing layers is performed in the previous section. It is assumed that the friction coefficient is not constant but varies in the transverse direction. It is seen from the comparison of Figs. 5.1 and 5.2 that non-uniform friction (presence of vegetated layers in the flow) results in larger growth rates than the case of uniform friction. The next question to answer is how this fact affects development of instability above the threshold. It is known from the previous studies on shallow water flows [30], [34], [46], that weakly nonlinear models can provide some insight into the development of instability in the case where the bed-friction number S is slightly smaller than the critical value (that is, the flow is linearly unstable but the growth rate of the most unstable mode is very small). Using the method of multiple scales an amplitude evolution equation is obtained following the procedure described in [62].

Consider the system of shallow water equations under the rigid-lid assumption (1.1)-(1.3). Introducing the stream function ψ by the relations (2.1), and eliminating the pressure from (1.1) - (1.3), we obtain equation (5.2). Next is the derivation of the equations of the second and third approximations, as in Chapter 2.2. Then only one new term we must add compared with equation (2.6):

$$
\frac{c_{\mathrm{f}_y}}{2h}\psi_y\sqrt{\psi_x^2+\psi_y^2}=\frac{c_{\mathrm{f}_y}}{2h}\begin{pmatrix}\psi^2{}_{0y}+2\varepsilon\psi_{0y}\psi_{1y}\\ +\varepsilon^2\left(2\psi_{0y}\psi_{2y}+1/2\psi_{1x}^2++\psi^2{}_{1y}-\psi^2{}_{0y}\right)\\ +\varepsilon^3\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}+2\psi_{3y}\psi_{0y}-2\psi_{0y}\psi_{1y}\right)\end{pmatrix}
$$

Introducing all expressions into (5.2) we obtain the following equation:

$$
\varepsilon\left(\psi_{1xxt}+\psi_{1yyt}\right)+\varepsilon^2\begin{pmatrix}\psi_{2xxt}+c_g\left(\psi_{1yy\xi}-\psi_{1xx\xi}\right)\\ +2\psi_{1x\xi\ t}+\psi_{2yyt}\end{pmatrix}+\varepsilon^3\begin{pmatrix}\psi_{3xxt}+\psi_{3yyt}+\psi_{1xx\tau}\\ +\psi_{1yy\tau}+2\psi_{2x\xi\ t}+\psi_{1\xi\xi\ t}\\ -c_g\left(\psi_{2xx\xi}+2\psi_{1x\xi\xi}+\psi_{2yy\xi}\right)\end{pmatrix}
$$

$$
\begin{aligned}
&+\varepsilon(\psi_{0y}\psi_{1xxx}+\psi_{0y}\psi_{1yyx}-\psi_{1x}\psi_{0yyy})\\
&+\varepsilon^2\begin{pmatrix}\psi_{0y}\psi_{2xxx}+3\psi_{0y}\psi_{1xx\xi}+\psi_{1y}\psi_{1xxx}+\psi_{1y}\psi_{1yyx}+\psi_{0y}\psi_{2yyx}\\ +\psi_{0y}\psi_{1\xi\ yy}-\psi_{1x}\psi_{1xxy}-\psi_{1x}\psi_{1yyy}-\psi_{2x}\psi_{0yyy}-\psi_{1\xi}\psi_{0yyy}\end{pmatrix}\\
&+\varepsilon^3\begin{pmatrix}\psi_{0y}\psi_{3xxx}+3\psi_{0y}\psi_{2xx\xi}+3\psi_{0y}\psi_{1x\xi\xi}+\psi_{1y}\psi_{2xxx}+3\psi_{1y}\psi_{1xx\xi}+\psi_{2y}\psi_{1xxx}\\ +\psi_{2y}\psi_{1yyx}+\psi_{1y}\psi_{2yyx}+\psi_{0y}\psi_{3yyx}+\psi_{1y}\psi_{1\xi\ yy}+\psi_{0y}\psi_{2\xi\ yy}-\psi_{2x}\psi_{1xxy}\\ -\psi_{1\xi}\psi_{1xxy}-\psi_{1x}\psi_{2xxy}-2\psi_{1x}\psi_{1xy\xi}-\psi_{1x}\psi_{2yyy}-\psi_{2x}\psi_{1yyy}-\psi_{3x}\psi_{0yyy}\\ -\psi_{1\xi}\psi_{1yyy}-\psi_{2\xi}\psi_{0yyy}\end{pmatrix}
\end{aligned}
\tag{5.17}
$$

$$
+\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}2\psi_{0y}\psi_{0yy}\\ +\varepsilon\left(\psi_{1xx}\psi_{0y}+2\psi_{0yy}\psi_{1y}+2\psi_{0y}\psi_{1yy}\right)\\ +\varepsilon^2\begin{pmatrix}\psi_{1xx}\psi_{1y}+\psi_{2xx}\psi_{0y}+2\psi_{2x\xi}\psi_{0y}+2\psi_{0yy}\psi_{2y}\\ +2\psi_{1yy}\psi_{1y}-2\psi_{0y}\psi_{0yy}+2\psi_{0y}\psi_{2yy}+2\psi_{1x}\psi_{1xy}\end{pmatrix}\\ +\varepsilon^3\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2\psi_{0y}}+\psi_{2xx}\psi_{1y}+2\psi_{1x\xi}\psi_{1y}+2\psi_{0y}\psi_{2x\xi}\\ +\psi_{1\xi\xi}\psi_{0y}+2\psi_{1yy}\psi_{2y}-\psi_{1xx}\psi_{0y}+2\psi_{2yy}\psi_{1y}-2\psi_{0yy}\psi_{1y}\\ -2\psi_{0y}\psi_{1yy}+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi y}+2\psi_{2x}\psi_{1xy}\\ +2\psi_{1\xi}\psi_{1xy}+2\psi_{3yy}\psi_{0y}+\psi_{3xx}\psi_{0y}+2\psi_{3y}\psi_{0yy}\end{pmatrix}\end{pmatrix}
$$

$$
+\frac{c_{\mathrm{f}\,y}}{2h}\begin{pmatrix}\psi_{0y}^2+2\varepsilon\psi_{0y}\psi_{1y}+\varepsilon^2\left(2\psi_{0y}\psi_{2y}+1/2\psi_{1x}^2++\psi^2{}_{1y}-\psi^2{}_{0y}\right)\\ +\varepsilon^3\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}+2\psi_{3y}\psi_{0y}-2\psi_{0y}\psi_{1y}\right)\end{pmatrix}=0\,.
$$

Collecting the terms of order ε , ε^2 , ε^3 we obtain the following expressions:

$$\varepsilon: \quad \begin{aligned}&\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}\psi_{1xxx}+\psi_{0y}\psi_{1yyx}-\psi_{1x}\psi_{0yyy}\\&+\frac{c_{\mathrm{f}}}{2h}\left(\psi_{1xx}\psi_{0y}+2\psi_{0yy}\psi_{1y}+2\psi_{0y}\psi_{1yy}\right)+\frac{c_{\mathrm{f}\,y}}{h}\psi_{0y}\psi_{1y}=0\ .\end{aligned} \tag{5.18}$$

$\varepsilon^2:$

$$\begin{aligned}&\psi_{2xxt}-c_g\psi_{1xx\xi}+2\psi_{1x\xi\,t}+\psi_{2yyt}-c_g\psi_{1yy\xi}+\psi_{0y}\psi_{2xxx}+3\psi_{0y}\psi_{1xx\xi}+\psi_{1y}\psi_{1xxx}\\&+\psi_{1y}\psi_{1yyx}+\psi_{0y}\psi_{2yyx}+\psi_{0y}\psi_{1\xi\,yy}-\psi_{1x}\psi_{1xxy}-\psi_{1x}\psi_{1yyy}-\psi_{2x}\psi_{0yyy}-\psi_{1\xi}\psi_{0yyy}\\&+\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{1y}+\psi_{2xx}\psi_{0y}+2\psi_{1x\xi}\psi_{0y}+\psi_{0yy}\psi_{2y}+2\psi_{1yy}\psi_{1y}+\psi_{2yy}\psi_{0y}\\-\psi_{0y}\psi_{0yy}+\psi_{2y}\psi_{0yy}+\psi_{0y}\psi_{2yy}+2\psi_{1x}\psi_{1xy}-\psi_{0y}\psi_{0yy}\end{pmatrix}\\&+\frac{c_{\mathrm{f}_y}}{2h}\left(2\psi_{0y}\psi_{2y}+1/2\psi_{1x}^2+\psi_{1y}^2-\psi_{0y}^2\right)=0.\end{aligned} \tag{5.19}$$

$\varepsilon^3:$

$$\begin{aligned}&\psi_{3xxt}-c_g\psi_{2xx\xi}+\psi_{1xx\tau}+2\psi_{2x\xi t}-2c_g\psi_{1x\xi\xi}+\psi_{1\xi\xi\,t}+\psi_{3yyt}-c_g\psi_{2yy\xi}+\psi_{1yy\tau}\\&\psi_{0y}\psi_{3xxx}+3\psi_{0y}\psi_{2xx\xi}+3\psi_{0y}\psi_{1x\xi\xi}+\psi_{1y}\psi_{2xxx}+3\psi_{1y}\psi_{1xx\xi}+\psi_{2y}\psi_{1xxx}\\&+\psi_{2y}\psi_{1yyx}+\psi_{1y}\psi_{2yyx}+\psi_{0y}\psi_{3yyx}+\psi_{1y}\psi_{1\xi\,yy}+\psi_{0y}\psi_{2\xi\,yy}-\psi_{2x}\psi_{1xxy}-\psi_{1\xi}\psi_{1xxy}\\&-\psi_{1x}\psi_{2xxy}-2\psi_{1x}\psi_{1xy\xi}-\psi_{1x}\psi_{2yyy}-\psi_{2x}\psi_{1yyy}-\psi_{3x}\psi_{0yyy}-\psi_{1\xi}\psi_{1yyy}-\psi_{2\xi}\psi_{0yyy}\\&+\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2\psi_{0y}}+\psi_{2xx}\psi_{1y}+2\psi_{1x\xi}\psi_{1y}+2\psi_{0y}\psi_{2x\xi}+\psi_{1\xi\xi}\psi_{0y}\\+2\psi_{1yy}\psi_{2y}-\psi_{1xx}\psi_{0y}+2\psi_{2yy}\psi_{1y}-2\psi_{0yy}\psi_{1y}-2\psi_{0y}\psi_{1yy}+2\psi_{1x}\psi_{2xy}\\+2\psi_{1x}\psi_{1\xi\,y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}+2\psi_{3yy}\psi_{0y}+\psi_{3xx}\psi_{0y}+2\psi_{3y}\psi_{0yy}\end{pmatrix}\\&+\frac{c_{\mathrm{f}_y}}{2h}\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}+2\psi_{3y}\psi_{0y}-2\psi_{0y}\psi_{1y}\right)=0.\end{aligned} \tag{5.20}$$

Let

$$\begin{aligned}L\varphi\equiv{}&\varphi_{xxt}+\varphi_{yyt}+\varphi_{0y}\varphi_{xxx}+\varphi_{0y}\varphi_{yyx}-\varphi_{0yyy}\varphi_x\\&+\frac{c_{\mathrm{f}}}{2h}\left(\varphi_{0y}\varphi_{xx}+2\varphi_{0yy}\varphi_y+2\varphi_{0y}\varphi_{yy}\right)+\frac{c_{\mathrm{f}_y}}{h}\varphi_{0y}\varphi_y\end{aligned},$$

then

$$L\psi_1=0\,.$$

Note, then $U = \psi_{0y}$ we obtain the following equation:

$$\psi_{1xxt} + \psi_{1yyt} + U\psi_{1xxx} + U\psi_{1xyy} - U_{yy}\psi_{1x}$$
$$+ \frac{c_{\mathrm{f}}}{2h}\left(U\psi_{1xx} + 2U_y\psi_{1y} + 2U\psi_{1yy}\right) + \frac{c_{\mathrm{f}\,y}}{h}U\psi_{1y} = 0\,. \tag{5.21}$$

Collecting the terms of order ε^2 we obtain the following equation for the function ψ_2:

$$L\psi_2 = c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi\ t} - 3U\psi_{1xx\xi} - \psi_{1y}\psi_{1xxx}$$
$$- \psi_{1y}\psi_{1yyx} - U\psi_{1\xi\ yy} + \psi_{1x}\psi_{1xxy} + \psi_{1x}\psi_{1yyy} + U_{yy}\psi_{1\xi}$$
$$- \frac{c_{\mathrm{f}}}{2h}\left(\psi_{1xx}\psi_{1y} + 2U\psi_{1x\xi} + 2\psi_{1yy}\psi_{1y} - 2UU_y + 2\psi_{1x}\psi_{1xy}\right) \tag{5.22}$$
$$- \frac{c_{\mathrm{f}\,y}}{2h}\left(1/2\psi_{1x}^2 + \psi_{1y}^2 - U^2\right).$$

Note that the operator L on the left-hand side of (5.22) is the same as in (5.21) and it will be the same for all orders of ε.

First we solve linear stability problem - the solution of the equation (5.21) will be sought in the form (2.5).

Substituting derivatives into the equation and simplifying we obtain:

$$\left(Uk - ck - i\gamma SU\right)\varphi_1'' - i\left(\gamma SU_y + \gamma_y SU\right)\varphi_1' + (k^3c - k^3U - kU_{yy} + ik^2\gamma\frac{S}{2}U)\varphi_1 = 0 \tag{5.23}$$

The boundary conditions are

$$\varphi_1(\pm\infty) = 0. \tag{5.24}$$

Details of the numerical solution of (5.23), (5.24) can be found in Section 2.1. We can find the critical values of the S_{c}, k_{c} and c_{c} (stability parameter, wave number and wave speed, respectively).

Assume now ψ_1 in the form (2.22).

Next, we consider the solution of (5.22). Substitute the derivatives in the right-hand side of the equation and simplify:

$$
\frac{c_{\mathrm{f}}}{2h}2UU_y+\frac{c_{\mathrm{f}_y}}{2h}U^2
$$

$$
+AA^*\begin{pmatrix} ik\varphi_{1y}\varphi^*_{1yy}-ik\varphi^*{}_{1y}\varphi_{1yy}++ik\varphi_1\varphi^*_{1yyy}-ik\varphi^*{}_1\varphi_{1yyy} \\ +\frac{c_f}{2h}k^2\varphi_1\varphi^*_{1y}+\frac{c_f}{2h}k^2\varphi^*_1\varphi_{1y}-\frac{c_f}{2h}2\varphi_{1y}\varphi^*_{1yy}-\frac{c_f}{2h}2\varphi^*_{1y}\varphi_{1yy} \\ -\frac{c_f}{2h}2k^2\varphi_{1y}\varphi^*_1-\frac{c_f}{2h}2k^2\varphi^*_{1y}\varphi_1-\frac{c_{f_y}}{2h}\varphi_1\varphi^*_1-\frac{c_{f_y}}{2h}2\varphi_{1y}\varphi^*_{1y} \end{pmatrix}
$$

$$
+A_\xi \mathrm{e}^{ik(x-ct)}\begin{pmatrix} -c_gk^2\varphi_1+c_g\varphi_{1yy} \\ -2k^2c\varphi_1+3Uk^2\varphi_1 \\ -U\varphi_{1yy}+U_{yy}\varphi_1 \\ -\frac{c_{\mathrm{f}}}{2h}2Uik\varphi_1 \end{pmatrix}+A_\xi{}^*\mathrm{e}^{-ik(x-ct)}\begin{pmatrix} -c_gk^2\varphi^*_1+c_g\varphi^*_{1yy} \\ -2k^2c\varphi^*_1+3Uk^2\varphi^*_1 \\ -U\varphi^*_{1yy}+U_{yy}\varphi^*_1 \\ +\frac{c_{\mathrm{f}}}{2h}2Uik\varphi^*_1 \end{pmatrix}
$$

$$
+A^2\mathrm{e}^{2ik(x-ct)}\begin{pmatrix} -ik\varphi_{1y}\varphi_{1yy}+ik\varphi_1\varphi_{1yyy} \\ +\frac{c_{\mathrm{f}}}{2h}3k^2\varphi_1\varphi_{1y}-\frac{c_{\mathrm{f}}}{2h}2\varphi_{1y}\varphi_{1yy} \\ +\frac{c_{\mathrm{f}_y}}{2h}0.5k^2\varphi_1^2+\frac{c_{\mathrm{f}_y}}{2h}\varphi_1^2 \end{pmatrix}+A^{*2}\mathrm{e}^{-2ik(x-ct)}\begin{pmatrix} -ik\varphi^*_{1y}\varphi^*_{1yy}-ik\varphi^*_1\varphi^*_{1yyy} \\ +\frac{c_{\mathrm{f}}}{2h}3k^2\varphi^*_1\varphi^*_{1y}-\frac{c_{\mathrm{f}}}{2h}2\varphi^*_{1y}\varphi^*_{1yy} \\ -\frac{c_{\mathrm{f}_y}}{2h}0.5k^2\varphi_1^{*2}-\frac{c_{\mathrm{f}_y}}{2h}\varphi_1^{*2} \end{pmatrix}
$$

Terms proportional to AA^*:

$$
ik\left(\varphi_{1y}\varphi^*_{1yy}-\varphi^*_{1y}\varphi_{1yy}+\varphi_1\varphi^*_{1yyy}-\varphi^*_1\varphi_{1yyy}\right)
$$
$$
-\frac{c_{\mathrm{f}}}{2h}\left(k^2\varphi_1\varphi^*_{1y}+k^2\varphi^*_1\varphi_{1y}+2\varphi_{1y}\varphi^*_{1yy}+2\varphi^*_{1y}\varphi_{1yy}\right)-\frac{c_{\mathrm{f}_y}}{2h}\left(\varphi_1\varphi^*_1+2\varphi_{1y}\varphi^*_{1y}\right)
$$

Terms proportional to $A_\xi\cdot\mathrm{e}^{ik(x-ct)}$:

$$
\left(c_g-U\right)\varphi_{1yy}+\left(-c_gk^2-2k^2c+3Uk^2+U_{yy}-\frac{c_{\mathrm{f}}}{2h}2Uik\right)\varphi_1
$$

Terms proportional to $A^2\cdot\mathrm{e}^{2ik(x-ct)}$

$$
ik\left(\varphi_1\varphi_{1yyy}-\varphi_{1y}\varphi_{1yy}\right)-\frac{c_{\mathrm{f}}}{2h}\left(2\varphi_{1y}\varphi_{1yy}-3k^2\varphi_1\varphi_{1y}\right)+\frac{c_{\mathrm{f}_y}}{2h}\left(0.5k^2\varphi_1^2+\varphi_1^2\right)
$$

Thus, the function ψ_2 should contain three groups of terms. More precisely we seek the solution to (5.22) in the form (2.41).

Substituting the derivatives in the left-hand side of the equation we obtain:

$$A_\xi e^{ik(x-ct)}\begin{pmatrix} ik^3c_g\varphi_2^{(1)} - ikc_g\varphi_{2yy}^{(1)} - ik^3U\varphi_2^{(1)} + ikU\varphi_{2yy}^{(1)} - ikU_{yy}\varphi_2^{(1)} \\ -\dfrac{c_\mathrm{f}}{2h}\left(Uk^2\varphi_2^{(1)} - 2U_y\varphi_{2y}^{(1)} - 2U\varphi_{2yy}^{(1)}\right) + \dfrac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(1)} \end{pmatrix}$$

$$+ A^2e^{2ik(x-ct)}\begin{pmatrix} 8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)} \\ -\dfrac{c_\mathrm{f}}{2h}\left(4k^2U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right) + \dfrac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(2)} \end{pmatrix}$$

$$+ A^*{}_\xi e^{-ik(x-ct)}\begin{pmatrix} -ik^3c_g\varphi_2^{(1)*} + ikc_g\varphi_{2yy}^{(1)*} - ik^3U\varphi_2^{(1)*} - ikU\varphi_{2yy}^{(1)*} - ikU_{yy}\varphi_2^{(1)*} \\ -\dfrac{c_\mathrm{f}}{2h}\left(Uk^2\varphi_2^{(1)*} - 2U_y\varphi_{2y}^{(1)*} - 2U\varphi_{2yy}^{(1)*}\right) + \dfrac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(1)*} \end{pmatrix}$$

$$+ A^{*2}e^{-2ik(x-ct)}\begin{pmatrix} -8ik^3c\varphi_2^{(2)*} + 2ikc\varphi_{2yy}^{(2)*} - 8ik^3U\varphi_2^{(2)*} - 2ikU\varphi_{2yy}^{(2)*} \\ + 2ikU_{yy}\varphi_2^{(2)*} - \dfrac{c_\mathrm{f}}{2h}\left(4k^2U\varphi_2^{(2)*} - 2U_y\varphi_{2y}^{(2)*} - 2U\varphi_{2yy}^{(2)*}\right) \\ + \dfrac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(2)*} \end{pmatrix}$$

$$+ AA^*\left(\frac{c_\mathrm{f}}{2h}\left(2U_y\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right) + 2U\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{(0)*}\right)\right) + \frac{c_{\mathrm{f}_y}}{h}U\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right)\right)$$

Terms proportional to AA^*:

$$\frac{c_\mathrm{f}}{2h}\left(2U_y\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right) + 2U\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{(0)*}\right)\right) + \frac{c_{\mathrm{f}_y}}{h}U\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{(0)*}\right)$$

Terms proportional to $A_\xi \cdot e^{ik(x-ct)}$:

$$ik^3c_g\varphi_2^{(1)} - ikc_g\varphi_{2yy}^{(1)} - ik^3U\varphi_2^{(1)} + ikU\varphi_{2yy}^{(1)} - ikU_{yy}\varphi_2^{(1)}$$
$$-\frac{c_\mathrm{f}}{2h}\left(Uk^2\varphi_2^{(1)} - 2U_y\varphi_{2y}^{(1)} - 2U\varphi_{2yy}^{(1)}\right) + \frac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(1)}$$

Terms proportional to $A^2 \cdot e^{2ik(x-ct)}$

$$8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)}$$
$$-\frac{c_\mathrm{f}}{2h}\left(4k^2U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right) + \frac{c_{\mathrm{f}_y}}{h}U\varphi_{2y}^{(2)}$$

Collecting the terms proportional to AA^* yields in the left-hand side of the equation (5.22) and in the right-hand side we obtain the equation for $\varphi_2^{(0)}$ (using: $S=\frac{c_{f_0}b}{h}$, $\varphi_2^{(0)}=\varphi_2^{(0)*}$)

$$\begin{aligned}&2\gamma S\left(U_y\varphi_{2y}^{(0)}+U\varphi_{2yy}^{(0)}\right)+2\gamma_y SU=ik\left(\varphi_{1y}\varphi_{1yy}^*-\varphi_{1y}^*\varphi_{1yy}+\varphi_1\varphi_{1yyy}^*-\varphi_1^*\varphi_{1yyy}\right)\\&-\frac{\gamma S}{2}\left(k^2\varphi_1\varphi_{1y}^*+k^2\varphi_1^*\varphi_{1y}+2\varphi_{1y}\varphi_{1yy}^*+2\varphi_{1y}^*\varphi_{1yy}\right)-\frac{\gamma_y S}{2}\left(\varphi_1\varphi_1^*+2\varphi_{1y}\varphi_{1y}^*\right)\end{aligned}\tag{5.25}$$

The boundary conditions have the form

$$\varphi_2^{(0)}(\pm\infty)=0.\tag{5.26}$$

Similarly, collecting the terms proportional to $\mathrm{e}^{ik(x-ct)}$ on the left-hand and right-hand sides of equation (5.22) we obtain the equation for the function $\varphi_2^{(1)}$:

$$\begin{aligned}&\left(ikU-ikc+\gamma SU\right)\varphi_{2yy}^{(1)}+\left(\gamma SU_y+\gamma_y SU\right)\varphi_{2y}^{(1)}+\begin{pmatrix}ik^3c-ik^3U-ikU_{yy}\\-\frac{\gamma S}{2}Uk^2\end{pmatrix}\varphi_2^{(1)}\\&=\left(c_g-U\right)\varphi_{1yy}+\left(-c_gk^2-2k^2c+3Uk^2+U_{yy}-i\gamma SUk\right)\varphi_1\end{aligned}\tag{5.27}$$

$$\varphi_2^{(1)}(\pm\infty)=0.\tag{5.28}$$

The adjoint operator L^{a} and adjoint eigenfunction φ_1^{a} are defined by the relation

$$\int_{-\infty}^{+\infty}\varphi_1^{\mathrm{a}}\cdot L\varphi_1dy=\int_{-\infty}^{+\infty}\varphi_1\cdot L^{\mathrm{a}}\varphi_1^{\mathrm{a}}dy.\tag{5.29}$$

The left-hand side of (5.29) is equal to zero since $L\varphi_1=0$. Thus, the adjoint equation is defined by the formula

$$L^{\mathrm{a}}\varphi_1^{\mathrm{a}}=0\,.\tag{5.30}$$

Integrating the left-hand side of (5.29) and using the boundary conditions (5.24) we obtain:

$$\begin{aligned}&\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\left(\left(U-c-\frac{i}{k}\gamma SU\right)\varphi_1''-\frac{i}{k}\left(\gamma_y SU+\gamma SU_y\right)\varphi_1'+\left(k^2c-k^2U-U_{yy}+ikU\frac{\gamma S}{2}\right)\varphi_1\right)dy\\&=\int_{-\infty}^{\infty}\varphi_1\begin{pmatrix}\varphi_1^{\mathrm{a}\,\prime\prime}\left(U-c-\frac{i}{k}\gamma SU\right)+\varphi_1^{\mathrm{a}\,\prime}\left(2U_y-\frac{i}{k}\left(\gamma SU_y+\gamma_y SU\right)\right)\\+\varphi_1^{\mathrm{a}}\left(k^2c-k^2U+ikU\frac{\gamma S}{2}\right)\end{pmatrix}dy=\int_{-\infty}^{\infty}\varphi_1\cdot L^{\mathrm{a}}\varphi_1^{\mathrm{a}}dy.\end{aligned}$$

We obtain the adjoint operator in the form

$$L^{a}\varphi_1^{a} \equiv \varphi_{1yy}^{a}\left(U - c - \gamma SU\frac{i}{k}\right) + \varphi_{1y}^{a}\left(2U_y - \frac{i}{k}\left(\gamma SU_y + \gamma_y SU\right)\right)$$
$$+\varphi_1^{a}\left(k^2 c - k^2 U + \frac{ik}{2}\gamma SU\right) = 0. \qquad (5.31)$$

The boundary conditions are

$$\varphi_1^{a}(\pm\infty) = 0. \qquad (5.32)$$

The adjoint eigenfunction φ_1^{a} is the solution of the problem (5.31), (5.32).

Applying the solvability condition to (5.27) we obtain

$$\int_{-\infty}^{+\infty}\varphi_1^{a}\left((c_g - U)\varphi_{1yy} + (-k^2 c_g - 2k^2 c + 3k^2 U + U_{yy} - ikU\gamma S)\varphi_1\right)dy = 0. \qquad (5.33)$$

Equation (5.33) defines the group velocity:

$$c_g = \frac{\eta_1}{\eta}, \qquad (5.34)$$

where

$$\eta = \int_{-\infty}^{+\infty}\varphi_1^{a}(\varphi_{1yy} - k^2\varphi_1)dy, \qquad (5.35)$$

$$\eta_1 = \int_{-\infty}^{+\infty}\varphi_1^{a}\left(U\varphi_{1yy} + \left(2k^2 c - 3k^2 U - U_{yy} + ikU\gamma S\right)\varphi_1\right)dy \qquad (5.36)$$

Finally, collecting the terms proportional to $\mathrm{e}^{2ik(x-ct)}$ we obtain:

$$\left(2ikU - 2ikc + \gamma SU\right)\varphi_{2yy}^{(2)} + \left(\gamma SU_y + \gamma_y SU\right)\varphi_{2y}^{(2)} + \begin{pmatrix} 8ik^3 c - 8ik^3 U \\ -2ikU_{yy} - 2\gamma Sk^2 U \end{pmatrix}\varphi_2^{(2)}$$
$$= ik\left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) + \frac{\gamma S}{2}\left(3k^2\varphi_1\varphi_{1y} - 2\varphi_{1y}\varphi_{1yy}\right) + \frac{\gamma_y S}{2}\left(\frac{k^2}{2}\varphi_1^2 + \varphi_1^2\right) \qquad (5.37)$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm\infty) = 0. \qquad (5.38)$$

Solving three boundary value problems (5.25) - (5.26), (5.27 - (5.28) and (5.37) - (5.38) numerically we obtain the functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$. The function ψ_2 (the second order correction) is then given by (2.41).

Let us consider the solution at the third order in ε. Similarly, collecting the terms of order ε^3 we obtain the following equation for the function ψ_3:

$$
\begin{aligned}
L\psi_3 &= c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-\psi_{1xx\tau}-2\psi_{2x\xi\ t}+2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}-\psi_{1yy\tau}\\
&-3U\psi_{2xx\xi}-3U\psi_{1x\xi\xi}-\psi_{1y}\psi_{2xxx}-3\psi_{1y}\psi_{1xx\xi}-\psi_{2y}\psi_{1xxx}-\psi_{2y}\psi_{1yyx}\\
&-\psi_{1y}\psi_{2yyx}-\psi_{1y}\psi_{1\xi\ yy}-U\psi_{2\xi\ yy}+\psi_{2x}\psi_{1xxy}+\psi_{1\xi}\psi_{1xxy}+\psi_{1x}\psi_{2xxy}\\
&+2\psi_{1x}\psi_{1xy\xi}+\psi_{1x}\psi_{2yyy}+\psi_{2x}\psi_{1yyy}+\psi_{1\xi}\psi_{1yyy}+\psi_{2\xi}U_{yy}\\
&-\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2U}+\psi_{2xx}\psi_{1y}+2\psi_{1x\xi}\psi_{1y}+2U\psi_{2x\xi}\\ +U\psi_{1\xi\xi}+2\psi_{1yy}\psi_{2y}+2\psi_{2yy}\psi_{1y}-U\psi_{1xx}-2U_y\psi_{1y}\\ -2U\psi_{1yy}+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi\ y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}\end{pmatrix}\\
&-\frac{c_{\mathrm{f}\,y}}{2h}\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}-2U\psi_{1y}\right)
\end{aligned}
\tag{5.39}
$$

Equation (5.39) also has a solution if and only if the right-hand side of (5.39) is orthogonal to all eigenfunctions φ_1^{a} of the corresponding homogeneous adjoint problem. Applying the solvability condition to (5.39) we obtain:

$$
\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}L\psi_3\,dy=0\Rightarrow
$$

$$
\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\begin{pmatrix}c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-\psi_{1xx\tau}-2\psi_{2x\xi\ t}+2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}-\psi_{1yy\tau}\\ -3U\psi_{2xx\xi}-3U\psi_{1x\xi\xi}-\psi_{1y}\psi_{2xxx}-3\psi_{1y}\psi_{1xx\xi}-\psi_{2y}\psi_{1xxx}\\ -\psi_{2y}\psi_{1yyx}-\psi_{1y}\psi_{2yyx}-\psi_{1y}\psi_{1\xi\ yy}-U\psi_{2\xi\ yy}+\psi_{2x}\psi_{1xxy}+\psi_{1\xi}\psi_{1xxy}\\ +\psi_{1x}\psi_{2xxy}+2\psi_{1x}\psi_{1xy\xi}+\psi_{1x}\psi_{2yyy}+\psi_{2x}\psi_{1yyy}+\psi_{1\xi}\psi_{1yyy}+\psi_{2\xi}U_{yy}\\ -\dfrac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2U}+\psi_{2xx}\psi_{1y}+2\psi_{1x\xi}\psi_{1y}+2U\psi_{2x\xi}\\ +U\psi_{1\xi\xi}+2\psi_{1yy}\psi_{2y}+2\psi_{2yy}\psi_{1y}-U\psi_{1xx}-2U_y\psi_{1y}\\ -2U\psi_{1yy}+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi\ y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}\end{pmatrix}\\ -\dfrac{c_{\mathrm{f}\,y}}{2h}\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}-2U\psi_{1y}\right)\end{pmatrix}dy=0.
\tag{5.40}
$$

Comparing (5.40) with (2.60) we see that only the last term is added to (5.40):

$$\frac{c_{\mathrm{f}\,y}}{2h}\left(\psi_{1x}\psi_{2x}+\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}-2U\psi_{1y}\right)=$$

$$\frac{c_{\mathrm{f}\,y}}{2h}\left(\begin{array}{l}\left(A\varphi_1 ik\mathrm{e}^{ik(x-ct)}-ikA^*\varphi_1^*\mathrm{e}^{-ik(x-ct)}\right)\left(\begin{array}{l}A_\xi\varphi_2^{(1)}ik\mathrm{e}^{ik(x-ct)}+A^2\varphi_2^{(2)}2ik\mathrm{e}^{2ik(x-ct)}\\-A^*{}_\xi\varphi_2^{(1)}ik\mathrm{e}^{-ik(x-ct)}-A^{2*}\varphi_2^{(2)*}2ik\mathrm{e}^{-2ik(x-ct)}\end{array}\right)\\ \left(A\varphi_1 ik\mathrm{e}^{ik(x-ct)}-ikA^*\varphi_1^*\mathrm{e}^{-ik(x-ct)}\right)\left(\begin{array}{l}A_\xi A^*\varphi_2^{(0)}+AA_\xi{}^*\varphi_2^{(0)}+A_{\xi\xi}\varphi_2^{(1)}\mathrm{e}^{ik(x-ct)}\\+2AA_\xi\varphi_2^{(2)}\mathrm{e}^{2ik(x-ct)}+A_\xi A^*\varphi_2^{(0)*}+AA_\xi{}^*\varphi_2^{(0)*}\\+A^*{}_{\xi\xi}\varphi_2^{(1)*}(y)\mathrm{e}^{-ik(x-ct)}+2A^*A_\xi^*\varphi_2^{(2)*}\mathrm{e}^{-2ik(x-ct)}\end{array}\right)\\ +2\left(\begin{array}{l}AA^*\varphi_{2y}^{(0)}+A_\xi\varphi_{2y}^{(1)}\mathrm{e}^{ik(x-ct)}+A^2\varphi_{2y}^{(2)}\mathrm{e}^{2ik(x-ct)}\\+AA^*\varphi_{2y}^{(0)*}+A^*{}_\xi\varphi_{2y}^{(1)*}\mathrm{e}^{-ik(x-ct)}\\+A^{2*}\varphi_{2y}^{(2)*}\mathrm{e}^{-2ik(x-ct)}\end{array}\right)\left(A\varphi_{1y}\,\mathrm{e}^{ik(x-ct)}+A^*\varphi_{1y}^*\,\mathrm{e}^{-ik(x-ct)}\right)\\ -2U\left(A\varphi_{1y}\,\mathrm{e}^{ik(x-ct)}+A^*\varphi_{1y}^*\mathrm{e}^{-ik(x-ct)}\right)\end{array}\right)$$

Using (5.40) we collect the terms proportional to:

$A_\tau\cdot\mathrm{e}^{ik(x-ct)}$:

$$-\psi_{1xx\tau}-\psi_{1yy\tau}\Rightarrow\left(-k^2\varphi_1+\varphi_{1yy}\right)=-\left(\varphi_{1yy}-k^2\varphi_1\right)\tag{5.41}$$

$A\cdot\mathrm{e}^{ik(x-ct)}$:

$$\begin{aligned}&-\frac{c_\mathrm{f}}{2h}\left(-2U_y\psi_{1y}-2U\psi_{1yy}+U\psi_{1xx}\right)-\frac{c_{\mathrm{f}_y}}{2h}2U\psi_{1y}\Rightarrow\\&-\frac{c_\mathrm{f}}{2h}\left(2U_y\varphi_{1y}+2U\varphi_{1yy}-Uk^2\varphi_1\right)-\frac{c_{\mathrm{f}_y}}{2h}\left(2U\varphi_{1y}\right)\end{aligned}\tag{5.42}$$

$A_{\xi\xi}\cdot\mathrm{e}^{ik(x-ct)}$:

$$\begin{aligned}&c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-2\psi_{2x\xi\ t}+2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}-3U\psi_{2xx\xi}-3U\psi_{1x\xi\xi}\\&-U\psi_{2\xi\ yy}+\psi_{2\xi}U_{yy}-\frac{c_\mathrm{f}}{2h}\left(2U\psi_{2x\xi}+U\psi_{1\xi\xi}\right)\Rightarrow\\&c_g\left(-k^2\varphi_2^{(1)}+\varphi_{2yy}^{(1)}\right)-2k^2c\varphi_2^{(1)}+2c_gik\varphi_1+ikc\varphi_1+3Uk^2\varphi_2^{(1)}-3Uik\varphi_1\\&-U\varphi_{2yy}^{(1)}+U_{yy}\varphi_2^{(1)}-\frac{c_\mathrm{f}}{2h}\left(2Uik\varphi_2^{(1)}+U\varphi_1\right)\end{aligned}\tag{5.43}$$

$A|A|^2$:

$$
\begin{aligned}
&-\psi_{2y}\left(\psi_{1xxx}+\psi_{1yyx}\right)+\psi_{1x}\left(\psi_{2xxy}+2\psi_{1xy\xi}+\psi_{2yyy}\right)\\
&-\psi_{1y}\left(\psi_{2xxx}+3\psi_{1xx\xi}+\psi_{2yyx}+\psi_{1\xi\ yy}\right)+\psi_{2x}\left(\psi_{1xxy}+\psi_{1yyy}\right)+\psi_{1\xi}\left(\psi_{1xxy}+\psi_{1yyy}\right)\\
&-\frac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2U}+\psi_{2xx}\psi_{1y}+2\psi_{1\xi}\psi_{1y}+2\psi_{1yy}\psi_{2y}+2\psi_{2yy}\psi_{1y}\\+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}\end{pmatrix}\\
&-\frac{c_{\mathrm{f}\,y}}{2h}\left(\psi_{1x}\psi_{2x}++\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}\right)\Rightarrow\\
&6ik^3\varphi_2^{(2)}\varphi_{1y}^*-2ik\varphi_{1y}^*\varphi_{2yy}^{(2)}+3ik^3\varphi_1^*\varphi_{2y}^{(2)}+ik^3\varphi_1\left(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)}\right)+ik\varphi_{2y}^{(2)}\varphi_{1yy}^*\\
&-ik\varphi_{1yy}\left(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)}\right)-ik\varphi_1^*\varphi_{2yyy}^{(2)}+ik\varphi_1\left(\varphi_{2yyy}^{(0)}+\varphi_{2yyy}^{*(0)}\right)+2ik\varphi_{1yyy}^*\varphi_2^{(2)}\\
&-\frac{S}{2}\begin{pmatrix}-k^2\varphi_1(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})+3k^2\varphi_1^*\varphi_{2y}^{(2)}-\dfrac{3k^4}{2u_0}\varphi_1^2\varphi_1^*++2\varphi_{1yy}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})\\+2\varphi_{1yy}^*\varphi_{2y}^{(2)}+2\varphi_{1y}(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)})+2\varphi_{2yy}^{(2)}\varphi_{1y}^*\end{pmatrix}\\
&-\frac{c_{\mathrm{f}\,y}}{2h}\left(-2k\varphi_1\varphi_2^{(2)}+2\varphi_{1y}\varphi_{2y}^{(2)}\right).
\end{aligned}
\tag{5.44}
$$

We rewrite (5.40) in the form:

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\left(\psi_{1xx\tau}+\psi_{1yy\tau}\right)dy=\\
&=\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\begin{pmatrix}c_g(\psi_{2xx\xi}+\psi_{2yy\xi})-2\psi_{2x\xi\ t}+2c_g\psi_{1x\xi\xi}-\psi_{1\xi\xi\ t}\\-3U\psi_{2xx\xi}-3U\psi_{1x\xi\xi}-U\psi_{2\xi\ yy}+\psi_{2\xi}U_{yy}\\-\dfrac{c_{\mathrm{f}}}{2h}\left(2U\psi_{2x\xi}+U\psi_{1\xi\xi}\right)\end{pmatrix}dy\\
&-\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\left(\frac{c_{\mathrm{f}}}{2h}\left(-2U_y\psi_{1y}-2U\psi_{1yy}+U\psi_{1xx}\right)+\frac{c_{\mathrm{f}_y}}{2h}2U\psi_{1y}\right)dy\\
&+\int_{-\infty}^{\infty}\varphi_1^{\mathrm{a}}\begin{pmatrix}-\psi_{1y}\psi_{2xxx}-3\psi_{1y}\psi_{1xx\xi}-\psi_{2y}\psi_{1xxx}-\psi_{2y}\psi_{1yyx}-\psi_{1y}\psi_{2yyx}-\psi_{1y}\psi_{1\xi\ yy}\\+\psi_{2x}\psi_{1xxy}+\psi_{1\xi}\psi_{1xxy}+\psi_{1x}\psi_{2xxy}+2\psi_{1x}\psi_{1xy\xi}+\psi_{1x}\psi_{2yyy}+\psi_{2x}\psi_{1yyy}\\+\psi_{1\xi}\psi_{1yyy}-\dfrac{c_{\mathrm{f}}}{2h}\begin{pmatrix}\psi_{1xx}\psi_{2y}+\dfrac{3\psi_{1xx}\psi_{1x}^2}{2U}+\psi_{2xx}\psi_{1y}+2\psi_{1x\xi}\psi_{1y}\\+2\psi_{1yy}\psi_{2y}+2\psi_{2yy}\psi_{1y}\\+2\psi_{1x}\psi_{2xy}+2\psi_{1x}\psi_{1\xi\ y}+2\psi_{2x}\psi_{1xy}+2\psi_{1\xi}\psi_{1xy}\end{pmatrix}\\-\dfrac{c_{\mathrm{f}\,y}}{2h}\left(\psi_{1x}\psi_{2x}++\psi_{1x}\psi_{2\xi}+2\psi_{2y}\psi_{1y}\right)\end{pmatrix}dy.
\end{aligned}
\tag{5.45}
$$

Using (5.41) - (5.44) equation (5.45) is written in the form of an amplitude evolution equation for slowly varying amplitude function $A(\xi,\tau)$ of the form:

$$\eta A_\tau = \sigma_1 A + \delta_1 A_{\xi\xi} - \mu_1 |A|^2 A$$

or

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A. \tag{5.46}$$

Equation (5.46) is the complex Ginzburg-Landau equation with complex coefficients σ, δ and μ :

$$\sigma = \frac{\sigma_1}{\eta},\ \delta = \frac{\delta_1}{\eta},\ \mu = \frac{\mu_1}{\eta}. \tag{5.47}$$

where $\sigma = \sigma_\mathrm{r} + i\sigma_\mathrm{i}$, $\delta = \delta_\mathrm{r} + i\delta_\mathrm{i}$ and $\mu = \mu_\mathrm{r} + i\mu_\mathrm{i}$ are complex coefficients which can be computed using linearized characteristics of the flow.

Coefficients σ_1, δ_1 μ_1 and η are given by:

$$\sigma_1 = \int_{-\infty}^{+\infty} \varphi_1^\mathrm{a} \left(\frac{\gamma S}{2} \left(-k^2 U \varphi_1 + 2U_y \varphi_{1y} + 2U\varphi_{1yy} \right) - \gamma_y S U \varphi_{1y} \right) dy, \tag{5.48}$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^\mathrm{a} \begin{pmatrix} \left(c_g - U\right)\varphi_{2yy}^{(1)} \\ + \varphi_2^{(1)} \left(-k^2 c_g - 2k^2 c + 3k^2 U + U_{yy} - ik\gamma S U \right) \\ + \varphi_1 \left(2ikc_g + ikc - 3ikU - U \dfrac{\gamma S}{2} \right) \end{pmatrix} dy, \tag{5.49}$$

$$\mu_1 = -\int_{-\infty}^{+\infty} \varphi_1^\mathrm{a} \begin{pmatrix} 6ik^3 \varphi_2^{(2)} \varphi_{1y}^* - 2ik\varphi_{1y}^* \varphi_{2yy}^{(2)} + 3ik^3 \varphi_1^* \varphi_{2y}^{(2)} + ik^3 \varphi_1 (\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) \\ - ik\varphi_{1yy} (\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + ik\varphi_{2y}^{(2)} \varphi_{1yy}^* - ik\varphi_1^* \varphi_{2yyy}^{(2)} \\ + ik\varphi_1 (\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}) + 2ik\varphi_{1yyy}^* \varphi_2^{(2)} \\ - \dfrac{\gamma S}{2} \begin{pmatrix} -k^2 \varphi_1 (\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 3k^2 \varphi_1^* \varphi_{2y}^{(2)} - \dfrac{3k^4}{2u_0} \varphi_1^2 \varphi_1^* \\ + 2\varphi_{1yy} (\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 2\varphi_{1yy}^* \varphi_{2y}^{(2)} \\ + 2\varphi_{1y} (\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}) + 2\varphi_{2yy}^{(2)} \varphi_{1y}^* \end{pmatrix} \\ - \gamma_y S \left(-k\varphi_1 \varphi_2^{(2)} + \varphi_{1y} \varphi_{2y}^{(2)} \right) \end{pmatrix} dy, \tag{5.50}$$

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^{a} (\varphi_{1yy} - k^2 \varphi_1) dy \qquad (5.51)$$

Especially important in this case is the sign of the real part of μ (known as the Landau constant in the literature). The Landau constant had the “wrong sign” in [62] which means that finite amplitude saturation was not possible and higher order terms (with respect to A) quickly become important so that (5.46) can be used for a very short time (in other words, practical application of (5.46) is very limited). In contrast to [62] it is shown in [30], [34], [46] that for shallow water flows the Landau constant in (5.46) has the “right sign” so that (5.46) can be used (and was successfully used in [30], [34] [46]) in order to describe some important features of shallow wake flows.

Experimental data presented in [66] showed that coherent structures exist in shallow mixing layers adjacent to a porous layer. Since Ginzburg-Landau equation has a rich variety of solutions depending on the values of the coefficients [1] it would be quite interesting to see whether predictions based on the Ginzburg-Landau model (5.46) will match experimental observations.

6. NUMERICAL RESULTS

In this Chapter we consider numerical aspects of weakly nonlinear analysis performed in the previous Chapters. One of the fundamental questions to answer is: “When does weakly nonlinear theory is applicable”? It is clear from the discussion in Chapter 2 (Section 2.3 and Fig. 2.1) that weakly nonlinear approach can be used in a small neighbourhood of the critical point. Thus, we can apply the theory and compute the coefficients of the Ginzburg-Landau equation. However, is there a criterion which can be used to convince us that the Ginzburg-Landau model can adequately represent the dynamics of a fully nonlinear model at least at the initial stage of transition period when the base flow becomes linearly unstable? The answer to this question (at least partially) is given in the paper by Suslov and Paolocci [63]. They proposed a relatively simple criterion for determination whether the Ginzburg-Landau equation can be used to analyse the dynamics of a linearly unstable flow. The criterion is as follows: if growth rates of an unstable perturbation can be well approximated by a parabola in the whole range of unstable wave numbers then the Ginzburg-Landau equation can be used to analyse the dynamics of the flow (at least in the beginning of the nonlinear regime).

In order to test this assertion we computed growth rates for the range of unstable wave numbers for the following values of the parameters of the problem for stability of slightly curved shallow mixing layers for base flow profile (2.11). The results of calculations are shown in Fig. 6.1 ($S < S_c$) for c_i ($c = c_r + ic_i$, when $c_i > 0$). As can be seen from the figures, the curve representing growth rates and parabolic fit are almost indistinguishable. Thus, we conclude that the Ginzburg-Landau equation can be successfully used to analyse the dynamics of the flow above the threshold.

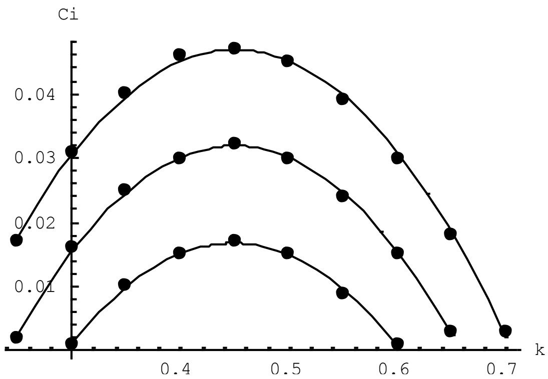

Fig. 6.1. Quadratic approximations of the growth rates for the following values of the parameters *S*=0.09, 0.08, 0.07 and 1/*R*=0.03 (from top to bottom).

Here we present the results of numerical calculations (Table 6.1) of the coefficients of the Ginzburg-Landau equation (2.66) using formulas (2.67) - (2.71). The results are shown for the base flow profile (2.11) (see Fig. 2.1) for the values of 1/R in the range from 0 to 0.04.

Table 6.1. Linear and Weakly Nonlinear Stability Characteristics for Different Values of 1/R (Chapter 2) and Base Flow Profile (2.11).

1/R	0.00	0.01	0.02	0.04
k	0.456	0.453	0.449	0.440
S	0.123	0.116	0.108	0.094
c	1.954	1.965	1.977	2.004
σ	$0.184 – 0.016i$	$0.173 – 0.015i$	$0.163 –0.013i$	$0.141 – 0.009i$
μ	$2.861 + 0.494i$	$3.046 + 0.539i$	$3.244+ 0.590i$	$3.673 + 0.720i$
c_g	1.927	1.924	1.922	1.914
δ	$6.487+13.238i$	$6.014+13.757i$	$5.472+14.447i$	$4.124+16.524i$

We also present here the calculations in a weakly nonlinear regime for the case of the problem considered in Chapter 5 (the case of non-uniform friction). The following "shape" profile $\gamma(y)$ is used to model non-uniform friction (see formula (5.1)):

$$\gamma(y) = \frac{\beta+1}{2} + \frac{\beta-1}{2}\tanh(\lambda y) .$$

The results of the numerical computations of the linear stability characteristics and the coefficients of the Ginzburg-Landau equation are shown in Table 6.2 below.

Table 6.2. Linear and Weakly Nonlinear Calculations for $\beta = 0.3$.

λ	0.25	0.5	1.0	1.5
k	0.442	0.437	0.438	0.437
S	0.198	0.205	0.211	0.214
c	1.972	1.985	2.004	2.018
σ	$0.195 – 0.487i$	$0.195 – 0.080i$	$0.183 – 0.133i$	$0.174 – 0.173i$
μ	$2.374 + 0.690i$	$2.151 + 0.687i$	$2.090 + 0.516i$	$2.092 + 0.262i$
c_g	1.956	1.981	2.007	2.018
δ	$7.077+13.243i$	$7.330 +12.434i$	7.403+10.752i	$7.255 + 9.645i$

After rescaling [1], the equation (5.46) for the complex amplitude $\tilde{A}$ has a form:

$$\frac{\partial\tilde{A}}{\partial\tilde{\tau}} = \tilde{A} + (1 + c_1 i)\frac{\partial^2\tilde{A}}{\partial\tilde{\xi}^2} - (1 + c_2 i)|\,\tilde{A}\,|^2\;\tilde{A} \quad , \tag{6.1}$$

where

$$\tilde{\tau} = \tau\sigma_{\mathrm{r}}\,, \qquad \tilde{\xi} = \xi\sqrt{\frac{\sigma_{\mathrm{r}}}{\delta_{\mathrm{r}}}}\,, \quad \tilde{A} = A\sqrt{\frac{\mu_{\mathrm{r}}}{\sigma_{\mathrm{r}}}}\exp(-ic_0\sigma_{\mathrm{r}}\tau),$$

$$c_0 = \frac{\sigma_{\mathrm{i}}}{\sigma_{\mathrm{r}}}\,, \qquad c_1 = \frac{\delta_{\mathrm{i}}}{\delta_{\mathrm{r}}}\,, \qquad c_2 = \frac{\mu_{\mathrm{i}}}{\mu_{\mathrm{r}}}\,.$$

Some closed form solutions of (6.1) are known in the literature [1], [10]. One of the simplest solutions is the solution of the form

$$\tilde{A} = a_0\exp\left(iq\tilde{\xi} + i\omega\tilde{\tau}\right), \tag{6.2}$$

where

$$a_0 = \sqrt{1 - q^2}\,, \qquad \omega = -c_2 - (c_1 - c_2)q^2\,.$$

Stability of (6.2) can be investigated by assuming that [38]

$$\tilde{A} = \left(a_0 + \hat{a}\cdot\exp\left(ik\tilde{\xi} + \lambda\,\tilde{\tau}\right) + \hat{a}^*\cdot\exp\left(-ik\tilde{\xi} + \lambda\,\tilde{\tau}\right)\right)\cdot\exp\left(iq\tilde{\xi} + i\omega\tilde{\tau}\right), \tag{6.3}$$

where $\hat{a}$ and $\hat{a}^*$ denote the amplitudes of the small perturbations.

Substituting (6.3) into (6.1) we obtain equation for λ. For the case of small k the stability condition has the form:

$$1 + c_1c_2 > 0 \tag{6.4}$$

provided that q satisfies the inequality

$$q^2 < \frac{1 + c_1c_2}{2c_2^2} \tag{6.5}$$

Condition (6.4) is known as the Benjamin-Feir stability condition. If (6.4) is not satisfied, than plane wave solutions of the form (6.2) are unstable (and, therefore, cannot be observed in experiments).

Numerical solutions of the Ginzburg-Landau equation (6.1) are presented below for different values of the parameters c_1, c_2 and different initial conditions. The problem is formulated as follows: find the solution of (6.1) for the given boundary conditions:

$$\tilde{A}|_{\tilde{\xi}=0} = 0, \ \tilde{A}|_{\tilde{\xi}=L} = 0 \tag{6.6}$$

and the initial condition:

$$\tilde{A}|_{\tilde{\tau}=0} = f(\tilde{\xi}). \tag{6.7}$$

Method of lines implemented in Mathematica 5 is used for the numerical solution to problem (6.1), (6.6), (6.7).

Table 6.3 shows numerical values of the coefficients c_1 and c_2 for different values of λ. As can be seen from Table 6.3, condition (6.4) is satisfied for all cases considered.

Table 6.3. Numerical Values of the Coefficients c_1 and c_2 of the Ginzburg-Landau Equation for $\beta = 0.3$.

β	0.3000	0.3000	0.3000	0.3000
λ	0.25	0.5	1.0	1.5
$c_1 = \frac{\delta_i}{\delta_r}$	1.8713	1.6963	1.4524	1.3294
$c_2 = \mu_i / \mu_r$	0.2906	0.3194	0.2469	0.1252
$1 + c_1 c_2$	2.1619	2.0157	1.6993	1.4547

The first computation is performed for the case c_1= 1.3293 and c_2=0.1251. The values of these parameters are taken from Table 6.3. The function $f(\tilde{\xi})$ in (6.7) is assumed to be a small random noise of order 0.01. The results are shown in Fig. 6.3. Since the parameters of the problem satisfy (6.4) and (6.5) (in other words, are in the region of stability), the modulus of the amplitude reaches constant value.

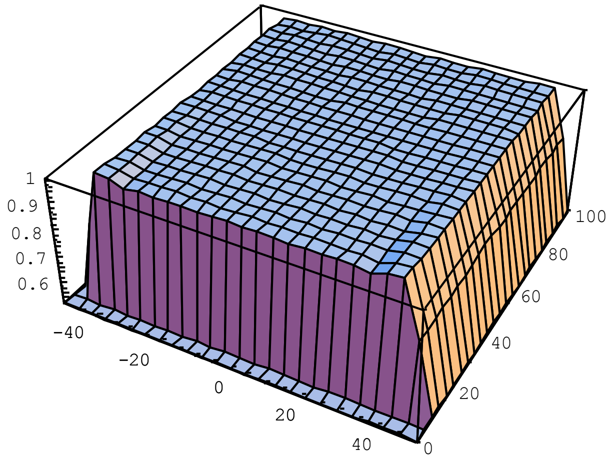

Fig. 6.3. Plot of the $|\tilde{A}|$.

The second set of computations corresponds is performed for the case $f(\tilde{\xi}) = \sqrt{1-q^2}\mathrm{e}^{iq\tilde{\xi}}$ where q = 0.5. The results are shown in Fig. 6.4.

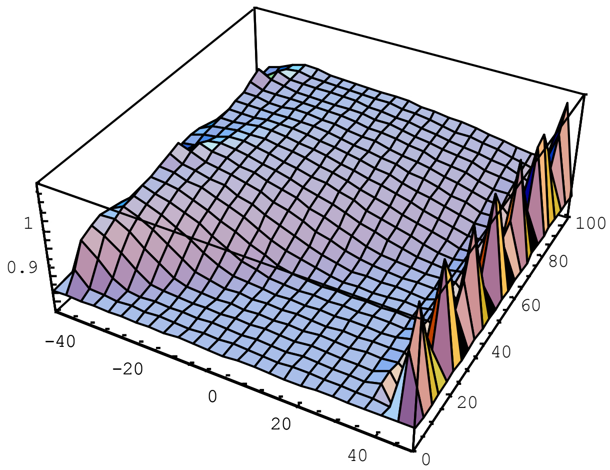

Fig. 6.4. Plot of the $|\tilde{A}|$.

Finally, we consider the case where the Benjamin - Feir stability condition (6.4) is not satisfied. The values of the parameters are taken from [46]: c_1 = -0.799564 and c_2 =2.189654 (these parameters correspond to weakly nonlinear analysis of wake flows). Random noise of order 0.01 is used as the initial condition. The results are shown in Figs. 6.5 and 6.6. As can be seen from the figure, stabilization of the amplitude does not occur in this case.

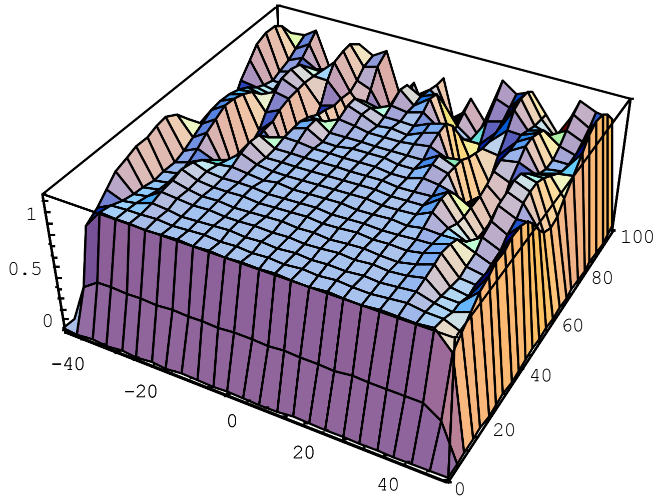

Fig. 6.5. Plot of the $\left|\tilde{A}\right|$.

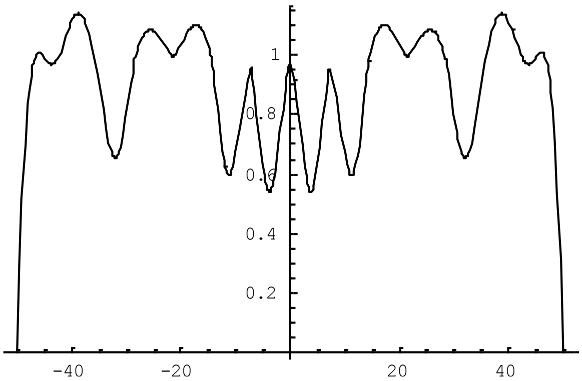

Fig. 6.6. The final configuration of $\left|\tilde{A}\right|$

These examples illustrate the well-known fact that the Ginzbureg-Landau model is quite rich in terms of different solutions. Illustrative computations in Figs. 6.3-6.6 show that both initial conditions and the values of the coefficients are responsible for spatio-temporal dynamics of the amplitude. The domain of applicability of the Ginzburg-Landau equation has to be defined. The equation is derived in a small neighbourhood of the critical point. Comparison of fully nonlinear simulations with predictions based on the Ginzburg-Landau model is required in order to test the validity of the model. This is left for future research.

CONCLUSION

The main conclusions from the linear stability analysis are as follows:

- Flow curvature effect is twofold: calculations show that the curvature gives a destabilizing effect on the unstable curved mixing layer and stabilizing effect on the stable curved mixing layer.

- Particle loading parameter has a stabilizing influence on the flow.

- Spatial stability analysis has been performed in the Thesis as well. One of the objectives has been to estimate the accuracy of Gaster's transformation away from the marginal stability curve.

- It is shown that the base flow asymmetry has a stabilizing influence on the flow.

- Calculations show that growth rates for the case of non-constant friction are higher than growth rates for the case of uniform friction.

Two methods of weakly nonlinear theory have been used in the Thesis for the stability analysis of shallow mixing layers. The first method uses parallel flow assumption. Using the method of multiple scales, the complex Ginzburg-Landau equation is derived from shallow water equations for slightly curved shallow water flow mixing layers, for two-component slightly curved mixing layers, for mixing layers with non-uniform friction. The coefficients of the equation are expressed in terms of integrals containing linearized characteristics of the flow.

The second method is based on the assumption that the wave length of perturbation is much smaller than the length scale of longitudinal evolution of the base flow. Perturbed stream function at the leading order is decomposed in this case into a slow-varying amplitude function and a fast-varying phase function. Solvability condition at the second order gives amplitude equation for the unknown amplitude of the most unstable mode.

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