RIGA TECHNICAL UNIVERSITY

FACULTY OF COMPUTER SCIENCE
AND INFORMATION TECHNOLOGY

Institute of Applied Mathematics

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SHALLOW FLOW STABILITY ANALYSIS WITH APPLICATIONS IN HYDRAULICS

Summary of the Doctoral Thesis

Submitted for the degree of Doctor of Mathematics
Subfield: Mathematical Modelling

Supervisor:
Prof., *Dr. math*. Andrejs Koliškins

RTU Press
Riga 2016

Eglīte I. **Shallow flow stability analysis with applications in hydraulics**.
Summary of the Doctoral Thesis. – R.: RTU Press, 2016, – 55 p.

Issued in accordance with the Decision of the RTU Department of Engineering Mathematics on 8 November 2016, Minutes No. 12501 – 1/4.

ISBN 978-9934-10-890-7

The Doctoral Thesis was elaborated at the Department of Engineering Mathematics, Institute of Applied Mathematics,
Faculty of Computer Science and Information Technology,
Riga Technical University from 2010 to 2015.

The research has been supported by the ESF within the Project "Support for the Implementation of Doctoral Studies at RTU" No. 2009/0144/1DP/1.1.2.1.2/09/IPIA/VIAA/00

The Thesis contains the introduction, 6 chapters, conclusion, and references.
Form of the thesis: dissertation in mathematics in the subfield of mathematical modelling

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The Thesis will be defended at the public session of the Doctoral Committee of Mathematics, University of Latvia on 16 December 2016 at 2:30 p.m., Riga, Zellu Street 25, University of Latvia, Faculty of Physics and Mathematics, Department of Mathematics, Room 233.

The Thesis is available at the Library of the University of Latvia, Riga, Kalpaka Blvd. 4

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Abstract

Linear and weakly nonlinear stability analysis of shallow mixing layers is presented in the Doctoral Thesis. The flow is assumed to be slightly curved along the longitudinal coordinate. Linear stability is analysed from temporal and spatial points of view under the rigid-lid assumption. The friction coefficient varies with respect to the transverse coordinate (the case of constant friction coefficient usually analysed in the literature is a particular case of the analysis presented in the Thesis). The corresponding linear stability problems are solved numerically using pseudo-spectral collocation method based on Chebyshev polynomials. In addition, the problem is generalized for the case of two-component shallow flows under the assumption of large Stokes numbers.

The effect of asymmetry of base flow profile on the stability characteristics is analysed. Two approaches to weakly nonlinear stability are presented as well. The first approach is based on the parallel flow assumption and can be applied for the case where the bed-friction number is slightly smaller than the critical value. Using the method of multiple scales, an amplitude evolution equation is derived for the most unstable mode. It is shown that for slightly curved shallow mixing layers, which contain or do not contain particles, the amplitude equation is the complex Ginzburg-Landau equation. The coefficients of the equation are calculated explicitly in terms of integrals containing linear stability characteristics of the flow. Stability of plane wave solutions of the Ginzburg-Landau equation is analysed. Numerical solutions of the Ginzburg-Landau equation are presented for different initial conditions.

The second approach takes into account slow longitudinal variation of the base flow. The analysis is based on weakly nonparallel *WKBJ* approximation. A first-order amplitude evolution equation is derived. The solution of the amplitude equation is then used to obtain the first-order approximation in the perturbation field.

Key words: Linear stability, weakly nonlinear theory, method of multiple scales, Ginzburg-Landau equation, collocation method.

Contents

Introduction

The Structure of the Thesis

The main goal of the Doctoral Thesis is to develop mathematical models, which can be used to analyse linear and weakly nonlinear instability of shallow mixing layers for the case of a single-component flow or two-component flow. The flow is assumed to be slightly curved along the longitudinal coordinate and the friction coefficient is assumed to be a function of the transverse coordinate.

The Thesis consists of the introduction, 6 chapters and conclusion. The volume of the Thesis is 104 pages. It is illustrated by 19 figures. Within the research, 69 reference sources have been consulted. The Doctoral Thesis has been written in English.

Chapter 1 presents a review of the literature used in the Doctoral Thesis. Basic equations used in the research are also described.

In Chapter 2, the linear stability and weakly nonlinear methods for analysis of slightly curved shallow mixing layers are presented in detail. Numerical methods used for the solution of stability problems are analysed.

Chapter 3 is devoted to the analysis of a similar problem for the case of slightly curved two-component shallow mixing layers. Linear and weakly nonlinear stability analysis is performed under the assumption of large Stokes numbers.

Chapter 4 is devoted to the spatial stability analysis of slightly curved shallow mixing layers.

In Chapter 5 analyses linear and weakly nonlinear instability of shallow mixing layers with variable friction in the transverse direction.

Chapter 6 is devoted to the numerical analysis of solution of Ginzburg-Landau equation.

The Topicality of the Research

The understanding of the interaction between fast and slow fluid streams in shallow mixing layers is important for the analysis of flows at river junctions and for design of compound channels. Real channels and rivers are not straight. Thus, the effect of curvature on the stability characteristics of shallow mixing layers should also be taken into account for proper design and analysis of compound channels. The case of non-uniform friction in the transverse direction is important from an environmental point of view. The friction coefficient in floodplain is usually higher than in the main channel (especially in case of floods). Complex vortex structures can accumulate contaminants and residues, thereby adversely affecting the environment. Hence, there is a need for a model that describes the shallow flow, as well as methods that allow analysing the flow stability and following up the development of perturbations.

The Objectives of the Doctoral Thesis

1. Analysis of linear and weakly nonlinear stability of slightly curved shallow mixing layers.
2. Investigation of linear and weakly nonlinear stability characteristics of slightly curved two-component shallow mixing layers.
3. Study of spatial stability of slightly curved shallow mixing layers.
4. Investigation of linear and weakly nonlinear instability of shallow mixing layers with variable friction.
5. Numerical analysis of linear and weakly nonlinear models.

Research Methodology

A base flow with a relatively simple structure is selected. Equations of motion are linearized in the neighbourhood of the base flow. The linearized equations are solved by the method of normal modes. The corresponding linear stability problems are solved numerically using pseudo-spectral collocation method based on Chebyshev polynomials.

Two approaches to weakly nonlinear stability analysis of single and two-component slightly curved shallow mixing layers are described. The first approach is based on the parallel flow assumption. Method of multiple scales is used in order to derive an amplitude evolution equation for the most unstable mode. It is shown that the amplitude equation is the complex Ginzburg-Landau equation. The second method takes into account a slow longitudinal variation of the base flow. The analysis is based on weakly nonparallel *WKBJ* approximation.

Scientific Novelty and Main Results

- Analysis of the asymmetry of the base flow velocity profile with regard to the linear stability.
- Investigation of the effect of flow curvature on the linear and weakly nonlinear stability.
- Linear and weakly nonlinear stability analysis of two-component shallow flows.
- Analysis of the two approaches to the solution of a linear stability problem:
    - spatial stability analysis;
    - temporal stability analysis.
- Analysis of the two approaches for weakly nonlinear stability:
    - the base flow does not depend on the longitudinal coordinate;
    - the base flow slightly varies on the longitudinal coordinate.
- Investigation of the stability of shallow mixing layers with variable friction in a linear and weakly nonlinear case.
- Explicit formulas for the computation of the coefficients of the Ginzburg-Landau equation are obtained for slightly curved shallow mixing layers, for

slightly curved two-phase shallow mixing layers, for shallow mixing layers with variable friction.
- Amplitude equation, describing the evolution of the amplitude of the perturbation with respect to the longitudinal coordinate, is derived.

Applications

Understanding stability characteristics and development of instability in shallow flows is important for design of compound channels. Since mixing layers also occur at river junctions and rivers are not straight, the analysis of the effect of curvature should also be taken into account. In some cases, flows can contain heavy particles moving with the fluid. Linear and weakly nonlinear analysis of two-component shallow mixing layers performed in the Thesis explained the effect of particle loading parameter on the stability characteristics of the flow under the assumption of large Stokes numbers.

Shallow water equations are nonlinear. Thus, numerical modelling of shallow water flows requires considerable computational resources since the number of parameters characterising the problem is large. Amplitude evolution equations for problems in thermal convection and Taylor-Couette flows are found to be quite useful in describing the dynamics of the corresponding flows at the initial stages of instability. Amplitude evolution equation in the form of a complex Ginzburg-Landau equation is derived in the Thesis from the equations of motion in a weakly nonlinear regime for the case of single or two-component slightly curved shallow mixing layers where the friction coefficient is constant or non-constant in the transverse direction. Since the Ginzburg-Landau equation is quite rich in terms of different solutions (depending on the values of the coefficients), in many cases it is used as a phenomenological equation for the analysis of spatio-temporal dynamics of complex flows.

The coefficients of the equation are estimated using experimental data, and the equation then can be used to model complex phenomena in fluid mechanics. It is shown in the Thesis that the coefficients of the Ginzburg-Landau equation can be calculated in closed form using linear stability characteristics of the flow. Thus, varying the parameters of the problem and re-calculating the coefficients of the Ginzburg-Landau equation one can use the equation to analyse spatio-temporal dynamics of the flow in a weakly nonlinear regime.

Publications

1. Eglite, I., Kolyshkin, A., and Ghidaoui, M. Weakly nonlinear analysis of shallow mixing layers with variable friction. In*: Materials of the 11th World Congress on Computational mechanics, 6th European Conference on Computational Fluid Dynamics*, Spain, Barcelona, 20–25 July 2014. Barcelona: CIMNE, 2014, pp. 1–2. ISBN 978-84-942844-7-2.

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Presentations at International Conferences

1. Weakly nonlinear analysis of shallow mixing layers with variable friction. *11th World Congress on Computational mechanics, 6th European Conference on Computational Fluid Dynamics*, Spain, Barcelona, 20–25 July 2014.

2. On the stability of shallow mixing layers with non-uniform friction. *The 10^th^ Latvian Mathematical Conference. The 2^nd^ International Conference on High Performance Computing and Mathematical Modelling*, Liepaja, Latvia, 11–12 April 2014.
3. Spatial stability analysis of shallow mixing layers with variable friction coefficient. *IASTED Intern. Conference on Modelling, Identification, and Control, MIC 2014*, Austria, Innsbruck, 17–19 February 2014.
4. Spatial and temporal instability of slightly-curved particle-laden shallow mixing layers. *V Intern. Conference on Computational Methods for Coupled Problems in Science and Eng.*, Spain, Ibiza, 17–19 June 2013.
5. Linear instability of shallow mixing layers with non-constant friction coefficient. *International Conference on Applied Mathematics and Scientific Computing*, Croatia, Šibenik, 10–14 June 2013.
6. Spatial stability analysis of curved shallow mixing layers. *15th International Conference on Mathematical Methods, Computational Techniques and Intelligent Systems*, Cyprus, Limassol, 21–23 March 2013.
7. Spatial instability of curved shallow mixing layers. *17th Intern. Conf. on Mathematical Modelling and Analysis,* Estonia, Tallinn, 6–9 June 2012.
8. Weakly nonlinear methods for stability analysis of slightly curved two-phase shallow mixing layers. *International Conference on Applied Mathematics and Sustainable Development: Special Track within SCET2012*, China, Xi'an, 27–30 May 2012.
9. Ginzburg-Landau model for curved two-phase shallow mixing layers. *ICCAM 2012: International Conference on Computational and Applied Mathematics*, Italy, Venice, 11–13 April 2012.
10. Asymptotic analysis of stability of slightly curved two-phase shallow mixing layers. *International Conference on Fluid Mechanics and Heat & Mass Transfer*, Greece, Corfu, 14–16 July 2011.
11. The effect of flow curvature on linear and weakly nonlinear instability of shallow mixing layers. *16th International Conference on Mathematical Modelling and Analysis*, Latvia, Sigulda, 25–28 May 2011.
12. The effect of slow variation of base flow profile on the stability of slightly curved mixing layers. *WASET International Conference*. Italy, Venice 27–29 April 2011.
13. Linear instability of curved shallow mixing layers. *The 6th IASME / WSEAS International Conference on Continuum Mechanics (CM'11)*, United Kingdom, Cambridge, 23–25 February 2011.
14. Spatial instability of asymmetric base flow profiles in shallow water. *15th International Conference Mathematical Modelling and Analysis,* Lithuania, Druskininkai, 26–29 May 2010.
15. The effect of asymmetry of base flow profile on the linear stability of shallow mixing layers. *10th WSEAS Intern. Conf. on Wavelet Analysis and Multirate Systems*, Tunisia, Kantaoui, Sousse, 3–6 May 2010.

Presentations at Local Conferences

1. Amplitūdas evolūcijas vienādojums stabilitātes analīzei divu fāžu sekla ūdens plūsmām. *9. Latvijas matemātikas konference.* Latvia, Jelgava, 30-31 March 2012.
2. Sekla ūdens plūsmas stabilitāte gadījumā, ja bāzes plūsmas profils nav simetrisks. 8. *Latvijas matemātikas konference.* Latvia, Valmiera, 9–10 April 2010.

1. Mathematical Formulation of the Problem

1.1. Literature Survey

Linear stability theory is widely used in order to analyse the behaviour of fluid flows ([9], [11], [49] and [55]). In many engineering applications of fluid mechanics, the transverse length scale of the flow is much larger than water depth. Such flows are usually referred to as "shallow flows". Curved shallow mixing layers are of a particular interest (flows in compound and composite channels and flows at river junctions represent typical examples of shallow mixing layers). Methods of analysis of shallow mixing layers include experimental investigation, numerical modelling and stability analysis [39]. Experimental investigation of shallow mixing layers is conducted in many papers (see, for example, [6], [59] and [60]). It is shown in these papers that bottom friction plays an important role in suppressing perturbations.

Linear stability analysis of shallow flows is performed in [5], [7], [32], [41], [43] and [52]. Rigid-lid assumption is used in [7] (water depth is assumed to be constant) to determine the critical values of the bed friction number for wake flows and mixing layers. The applicability of the rigid-lid assumption to the stability analyses of shallow flows is analysed in [32], where it is shown that for small Froude numbers the error in using the rigid-lid assumption is quite small. The effect of Froude number of the stability of shallow mixing layers in compound and composite channels is studied in [41]. Theoretical results and numerical computations presented in [5], [7], [32], [41], [43] and [52] confirm experimental observations: the bed friction number stabilizes the flow and reduces the growth of a mixing layer.

Centrifugal instability can also occur in shallow mixing layers. The effect of small curvature of the stability of free mixing layers is investigated in [34], [38] and [50]. It is shown in [50] that curvature has a stabilizing effect on a stably curved mixing layer and destabilizing effect on an unstably curved mixing layer.

Linear stability analysis can be used to determine how a particular flow becomes unstable. Critical values of the parameters (for example, critical bed friction number, critical wave number and so on) are also estimated from the linear stability theory. Development of instability above the threshold cannot be analysed by linear theory. Weakly nonlinear theories [35], [57] are used in order to construct an amplitude evolution equation for the most unstable mode. These theories are based on the method of multiple scales [40] and are applicable if the flow is unstable but the value of the parameter (for example, Reynolds number or bed friction number for shallow flows) is close to the critical value. In this case, the growth rate of unstable perturbation is small and one can hope to analyse the development of instability by means of relatively simple evolution equations. Such an approach is used in [57] for plane Poiseuille flow, in [2] and [46] in order to analyse instability of waves generated by wind and in [29], [32], [42], [43] and [51] for shallow wake flows. In fact, amplitude equations are used in the literature in two ways. First, a particular form of the evolution equation is selected a priori and the coefficients of the equation are estimated from experimental data. Then the equation with estimated coefficients is used to model the phenomenon of interest. Second, one can actually derive an evolution equation from the equations of motion. This approach is used in [2], [29], [43], [44], [47], [53], [56] and [57] where it is shown that for two-dimensional cases the evolution equation is the complex Ginzbrug-Landau equation.

Ginzburg-Landau equation and its properties are extensively studied in the literature ([1], [10]). Numerical analysis of the Ginzburg-Landau equation is simpler than numerical solution of the equations of motion. In addition, the analysis of stability of some simple (for example, periodic) solutions of the Ginzburg-Landau equation allows researchers to simplify the analysis of spatio-temporal dynamics of complex flows in fluid mechanics.

Linear instability of shallow mixing layers is analysed in [4], [7], [32] and [41] under the assumption that bottom friction is modelled by means of the Chezy formula [48] where the friction coefficient is assumed to be constant. Usually the friction coefficient is obtained from semi-empirical formulas [54], which relate the value of the friction coefficient to the Reynolds number of the flow and roughness of the surface. In such a case, the friction coefficient is assumed to be constant in the whole region of the flow.

In some applications, friction varies considerably in the transverse direction. One particular example is related to shallow flows under condition of partial vegetation. This situation often occurs during floods [61]. Friction force in a partially vegetated area is larger than in the main channel. It is shown in this case that the base flow profile is distorted and becomes asymmetric [61]. The difference in friction forces between partially vegetated area and the main channel is modelled in [61] by a step function. Linear stability analysis is conducted in [61] under the assumption that the base flow profile is symmetric.

1.2. Shallow Water Equations

Shallow water equations are depth-averaged equations, which are obtained by integrating equations of fluid mechanics with respect to the vertical coordinate. Since integration takes place over water depth it is necessary to specify stresses at the free surface and at the bottom. Stresses at the free surface are usually much smaller than the stresses at the bottom so that only bottom stresses are usually taken into account in shallow water equations. Empirical formulas (such as Chezy or Manning formulas) are used in practice in order to represent bottom friction. The detailed derivation of shallow water equations is given in [4].

Shallow water equations under the rigid-lid assumption in the presence of a small curvature have the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_{\mathrm{f}}}{2h}u\sqrt{u^2+v^2}-B(u^{\mathrm{p}}-u)=0, \tag{1.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{R}u^2+\frac{\partial p}{\partial y}+\frac{c_{\mathrm{f}}}{2h}v\sqrt{u^2+v^2}-B(v^{\mathrm{p}}-v)=0\,, \tag{1.3}$$

where x, y – geometric coordinates;

t – time;

u and v – the depth-averaged velocity components in the x and y directions;

p – the pressure;

h – water depth;

c_{f} – friction coefficient (can be constant or function of y);

B – particle loading parameter ([62], [63]);

u^{p} and v^{p} – the components of particle velocities;

$\frac{1}{R}=\frac{\delta_*}{R_*}<<1$ – small parameter;

R_* – the radius of curvature of the centreline of the curved mixing layer;

δ_* – the thickness of the mixing layer.

It is assumed in (1.2), (1.3) that the flow can contain heavy particles. The “lumped” effect of the particles is represented by the particle loading parameter B ([62], [63]). Equations (1)- (3) are written under the assumption of large Stokes number, which implies that there is no dynamic interaction between the particles and the carrier fluid.

Water surface in (1.1)- (1.3) is treated as the “rigid-lid”. Bottom friction in (1.2), (1.3) is modelled by means of the Chezy formula [4].

Friction coefficient in some applications varies in the transverse direction. Examples include shallow flows under conditions of partial vegetation during floods where water flows through partially vegetated area [61] or flows in compound and composite channels [41]. The variability of the friction coefficient in the transverse direction is modelled by a smooth differentiable shape function $c_{\mathrm{f}}(y)$.

2. Stability of Slightly Curved Shallow Mixing Layers

2.1. Linear Stability

Consider shallow water equations under the rigid-lid assumption in the presence of a small curvature in the form (1.1)-(1.3), where $B = 0$ ([14], [15], [16], [25]).

Eliminating the pressure p, then introducing the stream function $\psi(x, y, t)$ by the relations

$$u = \frac{\partial \psi}{\partial y} = \psi_y, \quad v = -\frac{\partial \psi}{\partial x} = -\psi_x \tag{2.1}$$

and using the notation $\Delta\psi = \psi_{xx} + \psi_{yy}$, we rewrite (1.1) – (1.3) in the form

$$\begin{aligned}(\Delta\psi)_t &+ \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2 + \psi_y^2} \\ &+ \frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2 + \psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) = 0,\end{aligned} \tag{2.2}$$

where the subscripts indicate the derivatives with respect to x, y and t.

Here the parallel flow assumption is used. The parallel flow assumption implies that the base flow does not change in the longitudinal direction. As pointed out in [47], this approximation is the leading-order solution in a multiple-scale expansion, which takes into account slow flow divergence.

Consider the stream function $\psi(x, y, t)$ of the form

$$\psi = \psi_0 + \psi', \tag{2.3}$$

where the quantity with prime represents small perturbations.

Substituting (2.3) into (2.2) and dropping the primes, we obtain the following equation:

$$\begin{aligned}&\psi_{xxt} + \psi_{yyt} + \psi_{0y}(\psi_{xxx} + \psi_{xyy}) - \psi_{0yyy}\psi_x \\ &+ \frac{c_{\mathrm{f}}}{2h}(\psi_{0y}\psi_{xx} + 2\psi_{0yy}\psi_y + 2\psi_{0y}\psi_{yy}) + \frac{2}{R}\psi_{0y}\psi_{xy} = 0\end{aligned}. \tag{2.4}$$

Following the method of normal modes [11], we assume a perturbation of the form

$$\psi(x, y, t) = \varphi(y)\mathrm{e}^{ik(x-ct)}, \tag{2.5}$$

where $\varphi(y)$ – the amplitude of normal perturbation;

k – wave number;

c – the phase speed of perturbation.

Substituting (2.5) and derivatives of ψ with respect to x, y or t into (2.4), denoted $U = \psi_{0y}$ (base flow), we obtain:

$$\varphi''\big(ik(U-c)+SU\big)+\varphi'\left(\frac{2}{R}ikU+SU_y\right)$$
$$+\varphi\left(ik^3c-ik^3U-ikU_{yy}-\frac{k^2SU}{2}\right)=0 \quad , \tag{2.6}$$

where $S=\dfrac{c_f\delta_*}{h}$ – bed-friction number;

δ_* – the width of the mixing layer.

The boundary conditions are

$$\varphi(\pm\infty)=0. \tag{2.7}$$

Using the linear stability theory, one can determine the conditions under which a particular flow becomes unstable. The linear stability of base flow determines the eigenvalues $c = c_r + ic_i$. The base flow is said to be linearly stable if all $c_i < 0$, and unstable, if at least one $c_i > 0$. Numerical solution of the corresponding eigenvalue problem (2.6) – (2.7) allows one to obtain the critical values of the parameters. However, the linear theory cannot be used to predict the evolution of the most unstable mode above the threshold. In the unstable region, perturbation grows exponentially with time (2.5). If the growth rate is large, then nonlinear effects quickly become dominant and there is little hope to analyse the development of instability analytically. However, if the growth rate of the unstable mode is relatively small, then weakly nonlinear theories can be used in order to develop an amplitude evolution equation for the most unstable mode.

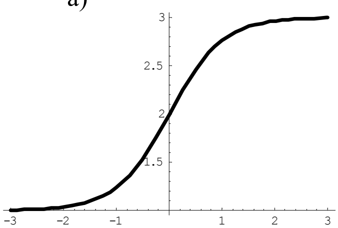
Base flows in the case of shallow water equations are usually chosen in the form of relatively simple model velocity profiles such as hyperbolic tangent profile for shallow mixing layers or hyperbolic secant profile for shallow wake flows. These profiles are chosen on the basis of careful analysis of available experimental data. The following two base flow profiles will be used (Fig. 2.1):

$$U(y)=2+\tanh y \tag{2.8}$$

and

$$U(y)=2-\tanh y\,. \tag{2.9}$$

a)



b)

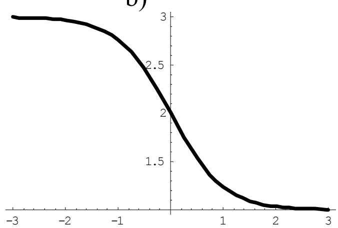


Fig. 2.1. Base flow profile a) $U(y)=2+\tanh y$ and b) $U(y)=2-\tanh y$.

Velocity profile (2.8) corresponds to stably curved mixing layer (in this case the high-speed stream is on the outside of the low-speed stream). Profile (2.9) represents the opposite situation (the high-speed stream is on the inside of the low-speed stream). It is shown in [52] that experimentally observed base flow velocity profile has similar shape to that of the plane mixing layer.

2.2. Numerical Method for Linear Stability

The pseudospectral collocation method [3] based on Chebyshev polynomials is used to solve eigenvalue problem (2.6) – (2.7) numerically. The interval $-\infty < y < +\infty$ is transformed into the interval (-1, 1) by means of the transformation $r = {}^2\!/_{\pi} \arctan y$. The solution to (2.6) is then sought in the form

$$\varphi(r) = \sum_{j=0}^{N-1} a_j (1-r^2) T_j(r), \qquad (2.10)$$

where $T_j(r) = \cos j \arccos r$ – the first kind Chebyshev polynomial of degree j;

a_j – unknown coefficients.

The factor $(1-r^2)$ guarantees that the boundary conditions (2.7) in terms of the new variable r are satisfied automatically at $r = \pm 1$. The use of those base functions considerably reduces the condition number [37].

The following set of collocation points is used to solve (2.6), (2.7):

$$r_m = \cos\frac{\pi m}{N+1}, \quad m = 1,2,...,N. \qquad (2.11)$$

Substituting the function $\varphi(r)$ and its derivatives at the collocation points into (2.6), we obtain the linear system of the equations in the form:

$$(B - cD)a = 0, \qquad (2.12)$$

where B and D are complex value matrices $N \times N$, matrix D is not singular and $a = (a_0\, a_1 ... a_{N-1})^{\mathrm{T}}$.

Problem (2.12) is solved numerically by means of the *IMSL* (*International Mathematics and Statistics Library*) routine *DGVCCG* that computes all of the eigenvalues and eigenvectors of a generalized complex eigensystem $Az = \lambda Bz$. The results of numerical computations for the case of stably curved shallow mixing layer (base flow velocity profile (2.8)) are shown in Table 2.1.

Table 2.1. The Results of Numerical Computations for the Case of Stably Curved Shallow Mixing Layer for Base Flow Velocity Profile (2.8)

k	$S(1/R=0)$	$S(1/R=0,01)$	$S(1/R=0.02)$	$S(1/R=0.03)$	$S(1/R=0.04)$
0.1	0.0260	0.0230	0.0205	0.0194	0.0258
0.2	0.0441	0.0408	0.0377	0.0348	0.0321
0.3	0.0554	0.0519	0.0485	0.0452	0.0421
0.4	0.0609	0.0572	0.0536	0.0501	0.0466
0.5	0.0612	0.0574	0.0536	0.0499	0.0462
0.6	0.0568	0.0529	0.0490	0.0451	0.0412
0.7	0.0482	0.0442	0.0402	0.0361	0.0322
0.8	0.0357	0.0316	0.0275	0.0234	0.0224
0.9	0.0196	0.0154	0.0150	0.0142	0.0138

Three marginal stability curves are shown in Fig. 2.2 for the three values of the parameter $1/R = 0,\ 0.02$ and 0.04, respectively (from top to bottom). The region of instability is below the curves [15].

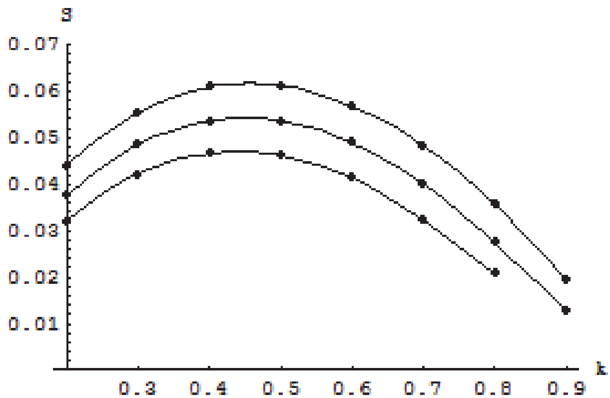


Fig. 2.2. Marginal stability curves for base flow profile (2.8).

Marginal stability curves for unstably curved shallow mixing layer (2.9) are shown in Fig. 2.3. The values of the parameter $1/R$ =0.04, 0.02 and 0, respectively (from top to bottom).

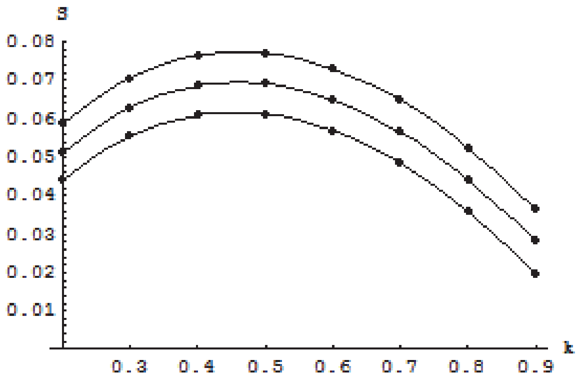


Fig. 2.3. Marginal stability curves for base flow profile (2.9).

Results of numerical computations show that the curvature stabilizes the flow in the case of stably curved mixing layer while for unstably curved mixing layer the curvature has a destabilizing effect on the flow [16].

2.3. Weakly Nonlinear Methods

Weakly nonlinear theories are usually constructed in the neighbourhood of a critical point (Fig. 2.4). Such equations are obtained in the past for the case of plane Poiseuille flow, shallow water flows, waves on the surface generated by wind and in some other situations (see [2], [29], [33], [43], [44], [46], [57]).

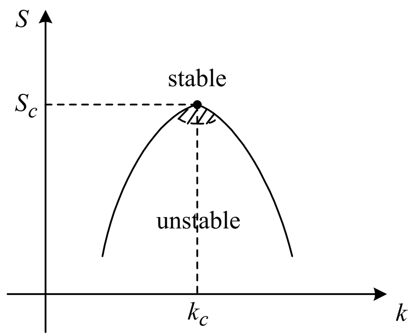


Fig. 2.4. A typical marginal stability curve for shallow water flow.

Suppose that S_c, k_c and c_c are the critical values of the stability parameter, wave number and wave speed, respectively. Then the most unstable mode (in accordance with the linear theory) is given by (2.5) with $S = S_c$, $k = k_c$ and $c = c_c$ where the eigenfunction $\varphi(y)$ can be replaced by $C\varphi(y)$. The constant C cannot be determined from the linear stability theory.

Consider a small neighbourhood of the critical point in the (k, S) plane where parameter S is assumed to be slightly below the critical value:

$$S = S_c(1-\varepsilon^2). \tag{2.13}$$

The parameter ε characterises how close the parameter S is to the critical value S_c. Moreover, (2.13) means that the base flow is unstable if the stability parameter is equal to S. However, since ε is small, the growth rate of the most unstable perturbations are also small. Therefore, one can try to describe analytically the development of instability using the weakly nonlinear theory.

Following the paper [57], we introduce the "slow" time τ and longitudinal coordinates ξ by the relations

$$\tau = \varepsilon^2 t,\ \xi = \varepsilon(x - c_g t), \tag{2.14}$$

where c_g is the group velocity.

The constant C in this case will be replaced by a slowly varying amplitude function A. Thus, $A = A(\xi, \tau)$ and the function ψ in (2.5) now has the form

$$\begin{aligned}\psi_1(x,\xi,y,t,\tau) &= A(\xi,\tau)\varphi(y)\mathrm{e}^{ik(x-ct)} + A^*(\xi,\tau)\varphi^*(y)\mathrm{e}^{-ik(x-ct)}\\ &= A(\xi,\tau)\varphi(y)\mathrm{e}^{ik(x-ct)} + c.c.\ ,\end{aligned} \tag{2.15}$$

where * and the abbreviation c.c. means the complex conjugate.

A perturbed solution ψ is sought in the form

$$\psi = \psi_0(y) + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + ..., \tag{2.16}$$

Substituting all expressions into (2.2) and collecting the terms of orders ε, ε^2, ε^3 we obtain three equations. Let

$$\begin{aligned}L\varphi \equiv\ & \varphi_{xxt} + \varphi_{yyt} + \varphi_{0y}\varphi_{xxx} + \varphi_{0y}\varphi_{yyx} - \varphi_{0yyy}\varphi_x\\ &+ \frac{2}{R}\varphi_{0y}\varphi_{1xy} + \frac{c_f}{2h}\left(\varphi_{0y}\varphi_{xx} + 2\varphi_{0yy}\varphi_y + 2\varphi_{0y}\varphi_{yy}\right),\end{aligned} \tag{2.17}$$

then the first equation (terms of orders ε) can be written as follows:

$$L\psi_1 = 0\ . \tag{2.18}$$

Using the notation $U = \psi_{0y}$, we rewrite (2.18) in the form

$$\begin{aligned}&\psi_{1xxt} + \psi_{1yyt} + U\psi_{1xxx} + U\psi_{1xyy} - U_{yy}\psi_{1x}\\ &+ \frac{2}{R}U\psi_{1xy} + \frac{c_f}{2h}\left(U\psi_{1xx} + 2U_y\psi_{1y} + 2U\psi_{1yy}\right) = 0\ .\end{aligned} \tag{2.19}$$

The second equation (terms of orders ε^2) is rewritten in the form

$$L\psi_2 = f_2\,, \tag{2.20}$$

where

$$\begin{aligned}
f_2 &= c_g(\psi_{1xx\xi}+\psi_{1yy\xi}) - 2\psi_{1x\xi\ t} - 3U\psi_{1xx\xi} - \psi_{1y}\psi_{1xxx} \\
&- \psi_{1y}\psi_{1yyx} - U\psi_{1\xi\ yy} + \psi_{1x}\psi_{1xxy} + \psi_{1x}\psi_{1yyy} + U_{yy}\psi_{1\xi} \\
&- \frac{c_\mathrm{f}}{2h}\left(\psi_{1xx}\psi_{1y} + 2U\psi_{1x\xi} + 2\psi_{1yy}\psi_{1y} - 2UU_y + 2\psi_{1x}\psi_{1xy}\right) \\
&- \frac{2}{R}\left(U\psi_{1\xi\ y} + \psi_{1y}\psi_{1xy}\right).
\end{aligned}$$

Note that the operator on the left-hand side of (2.20) is the same as in (2.18) and it will be the same for all orders of ε.

The third equation (terms of orders ε^3) can be rewritten as follows:

$$L\psi_3 = f_3 \tag{2.21}$$

where

$$\begin{aligned}
f_3 &= c_\mathrm{g}(\psi_{2xx\xi}+\psi_{2yy\xi}) - \psi_{1xx\tau} - 2\psi_{2x\xi\ t} + 2c_\mathrm{g}\psi_{1x\xi\xi} - \psi_{1\xi\xi\ t} \\
&- \psi_{1yy\tau} - 3U\psi_{2xx\xi} - 3U\psi_{1x\xi\xi} - \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \psi_{2y}\psi_{1xxx} \\
&- \psi_{2y}\psi_{1yyx} - \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi\ yy} - U\psi_{2\xi\ yy} + \psi_{2x}\psi_{1xxy} + \psi_{1\xi}\psi_{1xxy} \\
&+ \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} + \psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}U_{yy} \\
&- \frac{c_\mathrm{f}}{2h}\left(\begin{aligned}
&\psi_{1xx}\psi_{2y} + \frac{3\psi_{1xx}\psi_{1x}^2}{2U} + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2U\psi_{2x\xi} \\
&+ U\psi_{1\xi\xi} + 2\psi_{1yy}\psi_{2y} + 2\psi_{2yy}\psi_{1y} - U\psi_{1xx} - 2U_y\psi_{1y} \\
&- 2U\psi_{1yy} + 2\psi_{1x}\psi_{2xy} + 2\psi_{1x}\psi_{1\xi\ y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}
\end{aligned}\right) \\
&- \frac{2}{R}\left(U\psi_{2\xi\ y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi\ y} + \psi_{1xy}\psi_{2y}\right)
\end{aligned}$$

The solution to the linear stability problem (2.6), (2.7) is discussed in Section 2.2.

We consider the solution to (2.20). The following three groups of terms will emerge:

a) the terms that are independent on time;

b) the terms proportional to the first harmonic $\mathrm{e}^{ik(x-ct)}$ (here and in sequel we drop the subscripts and use the notation $k = k_\mathrm{c}$ and $c = c_\mathrm{c}$);

c) the terms proportional to the second harmonic $\mathrm{e}^{2ik(x-ct)}$.

Thus, the function ψ_2 should also contain the same three groups of terms. More precisely, we seek the solution to (2.20) in the form

$$\begin{aligned}\psi_2 &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)} + A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)} \\ &+ AA^*\varphi_2^{(0)*}(y) + A^*{}_\xi\varphi_2^{(1)*}(y)\mathrm{e}^{-ik(x-ct)} + A^{2*}\varphi_2^{(2)*}(y)\mathrm{e}^{-2ik(x-ct)} \\ &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)} + A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)} + c.c.,\end{aligned} \quad (2.22)$$

where $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y;

A^* denotes the complex conjugate of A;
the superscript reflects the index of the harmonic component;
the subscript represents the order of approximation.

Collecting the terms proportional to AA^* of the equation (2.20), we obtain the equation for $\varphi_2^{(0)}$:

$$\begin{aligned}&4S\left(U_y\varphi_{2y}^{(0)} + U\varphi_{2yy}^{(0)}\right) = ik\left(\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}\right) \\ &-\frac{S}{2}\left(k^2\varphi_1\varphi_{1y}^* + k^2\varphi_1^*\varphi_{1y} + 2\varphi_{1y}\varphi_{1yy}^* + 2\varphi_{1y}^*\varphi_{1yy}\right)\end{aligned}. \quad (2.23)$$

The boundary conditions are

$$\varphi_2^{(0)}(\pm\infty) = 0. \quad (2.24)$$

Similarly, collecting the terms proportional to $\mathrm{e}^{ik(x-ct)}$, we obtain the following equation for the function $\varphi_2^{(1)}$ with the boundary conditions:

$$\begin{aligned}&ik^3c\varphi_2^{(1)} - ikc\varphi_{2yy}^{(1)} - ik^3U\varphi_2^{(1)} + ikU\varphi_{2yy}^{(1)} - ikU_{yy}\varphi_2^{(1)} + ik\frac{2}{R}U\varphi_{2y}^{(1)} \\ &-\frac{c_\mathrm{f}}{2h}\left(Uk^2\varphi_2^{(1)} - 2U_y\varphi_{2y}^{(1)} - 2U\varphi_{2yy}^{(1)}\right) = \left(c_\mathrm{g} - U\right)\varphi_{1yy} \\ &-\frac{2}{R}U\varphi_{1y} + \left(-c_\mathrm{g}k^2 - 2k^2c + 3Uk^2 + U_{yy} - \frac{c_\mathrm{f}}{2h}2Uik\right)\varphi_1\end{aligned} \quad (2.25)$$

$$\varphi_2^{(1)}(\pm\infty) = 0. \quad (2.26)$$

Finally, collecting the terms proportional to $\mathrm{e}^{2ik(x-ct)}$ we obtain:

$$
\begin{gathered}
8ik^3 c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3 U\varphi_2^{(2)} + 2ikU\varphi_{2yy}^{(2)} - 2ikU_{yy}\varphi_2^{(2)} \\
+\frac{2}{R}2ikU\varphi_{2y}^{(2)} - \frac{c_\mathrm{f}}{2h}\left(4k^2 U\varphi_2^{(2)} - 2U_y\varphi_{2y}^{(2)} - 2U\varphi_{2yy}^{(2)}\right) \\
= ik\left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) - \frac{c_\mathrm{f}}{2h}\left(-3k^2\varphi_1\varphi_{1y} + 2\varphi_{1y}\varphi_{1yy}\right) - \frac{2}{R}ik\varphi_{1y}^2
\end{gathered}
\tag{2.27}
$$

with the boundary conditions

$$\varphi_2^{(2)}(\pm\infty) = 0. \tag{2.28}$$

Comparing (2.6) and (2.25), one can see that the left-hand sides are exactly the same if $\varphi_1(y)$ is replaced by $\varphi_2^{(1)}(y)$. Using the Fredholm alternative [64], we conclude that equation (2.25) has a solution if and only if the left-hand side is orthogonal to all eigenfunctions of the corresponding homogeneous adjoint problem.

The adjoint operator L^a and adjoint eigenfunction φ_1^a are defined by the relation

$$\int_{-\infty}^{+\infty} \varphi_1^\mathrm{a} \cdot L\varphi_1 dy = \int_{-\infty}^{+\infty} \varphi_1 \cdot L^\mathrm{a}\varphi_1^\mathrm{a} dy. \tag{2.29}$$

The left-hand side of (2.29) is equal to zero since $L\varphi_1 = 0$. Thus, the adjoint equation is defined by the formula

$$L^\mathrm{a}\varphi_1^\mathrm{a} = 0. \tag{2.30}$$

Integrating the left-hand side of (2.29) and using the boundary conditions (2.7), we obtain the adjoint operator

$$
\begin{gathered}
L^\mathrm{a}\varphi_1^\mathrm{a} \equiv \varphi_{1yy}^\mathrm{a}\left(U - c - \frac{iSU}{k}\right) + \varphi_{1y}^\mathrm{a}\left(2U_y - \frac{iSU_y}{k} - \frac{2}{R}U\right) \\
+\varphi_1^\mathrm{a}\left(k^2 c - k^2 U + \frac{ikSU}{2} - \frac{2}{R}U_y\right) = 0
\end{gathered}
\tag{2.31}
$$

The boundary conditions are

$$\varphi_1^\mathrm{a}(\pm\infty) = 0. \tag{2.32}$$

Applying the solvability condition to (2.25), we define the group velocity:

$$c_g = \frac{\eta_1}{\eta}, \tag{2.33}$$

where

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^a (\varphi_{1yy} - k^2 \varphi_1) dy, \tag{2.34}$$

$$\eta_1 = \int_{-\infty}^{+\infty} \varphi_1^a \left(U\varphi_{1yy} + \frac{2}{R} U\varphi_{1y} + (2k^2 c - 3k^2 u_0 - U_{yy} + ikUS)\varphi_1 \right) dy. \tag{2.35}$$

Solving three boundary value problems (2.23) – (2.24), (2.25) – (2.26) and (2.27) – (2.28) numerically, we obtain the functions $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$. The function ψ_2 is then given by (2.22).

Let us consider the solution at the third order in ε. Equation (2.21) also has a solution if and only if the right-hand side of (2.21) is orthogonal to all eigenfunctions φ_1^a of the corresponding homogeneous adjoint problem (2.31), (2.32). Applying the solvability condition to (2.21), we obtain:

$$\int_{-\infty}^{\infty} \varphi_1^a \cdot L\psi_3 dy = 0 \tag{2.36}$$

Equation (2.36) is converted to the amplitude evolution equation for slowly varying amplitude function $A(\xi, \tau)$ of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A. \tag{2.37}$$

Equation (2.37) is the complex Ginzburg-Landau equation with complex coefficients σ, δ and μ

$$\sigma = \frac{\sigma_1}{\eta}, \ \delta = \frac{\delta_1}{\eta}, \ \mu = \frac{\mu_1}{\eta}. \tag{2.38}$$

Coefficients $\sigma_1, \delta_1 \ \mu_1$ are given by

$$\mu_1 = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \begin{pmatrix} 6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)} + ik^3\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) \\ - ik\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + ik\varphi_{2y}^{(2)}\varphi_{1yy}^* + ik\varphi_1\left(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}\right) \\ - ik\varphi_1^*\varphi_{2yyy}^{(2)} + 2ik\varphi_{1yyy}^*\varphi_2^{(2)} - 2\frac{ik}{R}\left(\varphi_{2y}^{(2)}\varphi_{1y}^* + \varphi_{2y}^{(0)}\varphi_{1y}\right) \\ -\frac{S}{2}\begin{pmatrix} -k^2\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 3k^2\varphi_1^*\varphi_{2y}^{(2)} - \frac{3k^4}{2u_0}\varphi_1^2\varphi_1^* \\ + 2\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 2\varphi_{1yy}^*\varphi_{2y}^{(2)} \\ + 2\varphi_{1y}\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}\right) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^* \end{pmatrix} \end{pmatrix} dy\,, \tag{2.39}$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \begin{pmatrix} \left(c_{\mathrm{g}} - U\right)\varphi_{2yy}^{(1)} - 2\frac{U}{R}\varphi_{2y}^{(1)} \\ + \varphi_2^{(1)}\left(-k^2c_{\mathrm{g}} - 2k^2c + 3k^2U + U_{yy} - ikSU\right) \\ + \varphi_1\left(2ikc_{\mathrm{g}} + ikc - 3ikU - U\frac{S}{2}\right) \end{pmatrix} dy\,, \tag{2.40}$$

$$\sigma_1 = \frac{S}{2}\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}}\left(-k^2U\varphi_1 + 2U_y\varphi_{1y} + 2U\varphi_{1yy}\right)dy\,. \tag{2.41}$$

Formulas (2.38) – (2.41) represent the coefficients of equation (2.37) in terms of the characteristics of the linear stability of the flow.

2.4. Numerical Method for Weakly Nonlinear Stability

In this subsection, we propose a numerical method for the calculation of the coefficients of the Ginzburg-Landau equation. More precisely, in order to obtain σ, δ and μ we need to perform the following calculations:

1. Solve the linear stability problem (2.6) – (2.7) and determine the critical values of the parameters k, S, c and the corresponding eigenfunction $\varphi_1(y)$;

2. Solve the homogeneous adjoint problem (2.31) – (2.32) and determine the adjoint eigenfunction φ_1^{a};

3. Solve three boundary value problems (2.23) – (2.24), (2.25) – (2.26) and (2.27) – (2.28) and determine the functions $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$;

4. Evaluate the integrals in (2.38).

The elements of the matrices B and D (see (2.12)) can be computed and the generalized eigenvalue problem (2.6) – (2.7) can be solved numerically. Similar approach can be used in order to solve boundary value problems (2.23) – (2.24) and (2.27) – (2.28). System of linear algebraic equations of the form

$$Fa = G \tag{2.42}$$

is obtained in each case after discretization where $a = \left(a_0\, a_1 ... a_{N-1}\right)^{\mathrm{T}}$. The matrix F is not singular for these problems. Therefore, any linear equation solver can be used in order to find a. Thus, the functions $\varphi_2^{(0)}(y)$ and $\varphi_2^{(2)}(y)$ can be evaluated by means of the expansions of the form (2.10).

The same form of the expansion (2.10) is used to solve boundary value problem (2.25) – (2.26). Equation of the form (2.42) is also obtained after discretization in this case, but the matrix F is singular since the corresponding homogeneous part of (2.29) has a nontrivial solution at $S = S_{\mathrm{c}}$, $k = k_{\mathrm{c}}$ and $c = c_{\mathrm{c}}$. Equation (2.42) is solved in this case by means of the singular value decomposition method [36]. It is known that if F is a complex $N \times N$ matrix, then there are orthogonal $N \times N$ matrices U and V such that

$$U^{\mathrm{H}} \cdot F \cdot V = \Sigma, \tag{2.43}$$

where $\Sigma = diag(\gamma_1, \gamma_2, ..., \gamma_N)$.

Equation (2.43) is called the singular value decomposition of the matrix F and $\gamma_1, \gamma_{21}, ..., \gamma_N$ are the singular values of F. In our case, only the last of the singular values will be equal to zero ($\gamma_1 > \gamma_2 > ... > \gamma_{N-1} > \gamma_N = 0$). Hence, the solution to (2.42) in this case can be written in the form

$$a = V \cdot \Sigma^{-1} \cdot U^{\mathrm{H}} \cdot G, \tag{2.44}$$

where the last column of V, the last row of U^{H}, the last column and the last row of Σ^{-1} are deleted.

In component form, the solution is

$$a = \sum_{i=1}^{N-1} \frac{U_i^{\mathrm{H}} \cdot G \cdot V_i}{\gamma_i}, \tag{2.45}$$

where U^{H} and V_i are vectors (columns of the matrices U^{H} and V).

The values of the function $\varphi_2^{(1)}(y)$ can be computed using formula (2.10) where the coefficients a_j are the components of the vector a in (2.45).

The final step of the computational procedure involves the calculation of integrals in (2.38), which uses adaptive quadrature formula described in [30].

3. Linear and Weakly Nonlinear Instability of Slightly Curved Two-Component Shallow Mixing Layers

3.1. Linear Stability

It is assumed that the carrier fluid contains small heavy particles. The assumptions that are used in the derivation of the governing equations are summarised in Section 1.2. Introducing the stream function $\psi(x, y, t)$ we transform the shallow water equations to the form

$$(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2 + \psi_y^2}$$
$$+ \frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2 + \psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) + B\Delta\psi = 0. \qquad (3.1)$$

A perturbed solution to (3.1) is sought in the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + \ldots \qquad (3.2)$$

Substituting (3.2) into (3.1) and linearizing the resulting equation in the neighbourhood of the base flow, we obtain (see 2.1):

$$L\psi_1 = 0, \qquad (3.3)$$

where

$$L\psi \equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{0y}\psi_{yyx} - \psi_{0yyy}\psi_x + \frac{2}{R}\psi_{0y}\psi_{xy}$$
$$+ \frac{c_{\mathrm{f}}}{2h}(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}) + B(\psi_{1xx} + \psi_{1yy}) \qquad (3.4)$$

The solution to (3.3) is sought in the form of a normal mode [11]

$$\psi_1(x, y, t) = \varphi_1(y)\mathrm{e}^{ik(x-ct)}. \qquad (3.5)$$

Using (3.3) and (3.5), we obtain boundary value problem

$$L\varphi_1 \equiv \varphi_1''\left(U - c - \frac{iSU}{k} - \frac{iB}{k}\right) + \varphi_1'\left(\frac{2U}{R} - \frac{iSU_y}{k}\right)$$
$$+ \varphi_1\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2} + ikB\right) = 0 \qquad (3.6)$$

$$\varphi_1(\pm\infty) = 0. \qquad (3.7)$$

Problem (3.6), (3.7) is usually solved numerically. Details of numerical algorithm based on the collocation method are given in Section 2.2. Thus, solution (3.6), (3.7) allows one to obtain the critical values of the parameters S_c, k_c and c_c. A typical marginal stability curve for shallow water flows is a convex curve with one maximum (the coordinates of the maximum point in the (k, S) plane are $k = k_c$ and $S = S_c$).

3.2. Weakly Nonlinear Stability

Assume that the bed-friction number is slightly smaller than the critical value

$$S = S_c(1-\varepsilon^2). \tag{3.8}$$

Following [57], we introduce the following "slow" variables:

$$\tau = \varepsilon^2 t,\ \ \xi = \varepsilon(x - c_g t)\ . \tag{3.9}$$

The stream function ψ_1 in (3.5) is replaced by

$$\psi_1(x,y,t,\xi,\tau) = A(\xi,\tau)\varphi_1(y)\mathrm{e}^{ik(x-ct)}, \tag{3.10}$$

where $\varphi_1(y)$ is the eigenfunction with $S = S_c$, $k = k_c$ and $c = c_c$.

The objective is to derive equation for the evolution of the amplitude function $A(\xi,\tau)$. Using (3.1), (3.2), (3.11) and collecting the terms that contain ε^2 we obtain

$$L_1\psi_2 = \widetilde{f}_2\ . \tag{3.12}$$

Analysing the structure of the function $\widetilde{f}_2$ in (3.12), we conclude that ψ_2 should be sought in the form

$$\psi_2 = AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\mathrm{e}^{ik(x-ct)} + A^2\varphi_2^{(2)}(y)\mathrm{e}^{2ik(x-ct)}, \tag{3.13}$$

where $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y.

Substituting ψ_2 into (3.12) and collecting the time-independent terms, we obtain the boundary value problem:

$$\begin{aligned}&2S\left(U_y\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + U\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}\right)\right) + 2B\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}\right)\\ &= ik\begin{pmatrix}\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \\ + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}\end{pmatrix} - \frac{S}{2}\begin{pmatrix}k^2\left(\varphi_1\varphi_{1y}^* + \varphi_1^*\varphi_{1y}\right) + \\ + 2\left(\varphi_{1y}^*\varphi_{1yy} + \varphi_{1yy}^*\varphi_{1y}\right)\end{pmatrix}\end{aligned} \tag{3.14}$$

$$\varphi_2^{(0)}(\pm\infty) = 0. \tag{3.15}$$

Collecting the terms containing the first harmonic, we obtain the boundary value problem

$$\left(U - c - SU\frac{i}{k} - \frac{iB}{k}\right)\varphi^{(1)}_{2yy} + \left(2\frac{U}{R} - SU_y\frac{i}{k}\right)\varphi^{(1)}_{2y}$$

$$+\left(k^2c - k^2U - U_{yy} + \frac{ikSU}{2} + ikB\right)\varphi^{(1)}_2 = -\frac{i}{k}\left(c_g - U\right)\varphi_{1yy} \quad (3.16)$$

$$+2\frac{iU}{kR}\varphi_{1y} + \left(2ikc - 3ikU - \frac{i}{k}U_{yy} + ikc_g - US - 2B\right)\varphi_1$$

$$\varphi^{(1)}_2(\pm\infty) = 0. \quad (3.17)$$

Finally, collecting the terms that contain the second harmonic, we obtain

$$8ik^3c\varphi^{(2)}_2 - 2ikc\varphi^{(2)}_{2yy} - 8ik^3U\varphi^{(2)}_2 + 2ikU\varphi^{(2)}_{2yy} - 2ikU_{yy}\varphi^{(2)}_2$$

$$+S\begin{pmatrix} 2U_y\varphi^{(2)}_{2y} + 2U\varphi^{(2)}_{2yy} \\ -4k^2U\varphi^{(2)}_2 \end{pmatrix} + \frac{4ik}{R}U\varphi^{(2)}_{2y} + B\left(\varphi^{(2)}_{2yy} - 4k^2\varphi^{(2)}_2\right) \quad (3.18)$$

$$= ik\left(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}\right) - S\left(2\varphi_{1y}\varphi_{1yy} - 3k^2\varphi_1\varphi_{1y}\right) - \frac{2ik}{R}\varphi^2_{1y}$$

$$\varphi^{(2)}_2(\pm\infty) = 0. \quad (3.19)$$

The adjoint operator L^{a} and adjoint eigenfunction φ^{a}_1 are defined as follows:

$$\int_{-\infty}^{+\infty}\varphi^{\mathrm{a}}_1 \cdot L\varphi_1 dy = \int_{-\infty}^{+\infty}\varphi_1 \cdot L^{\mathrm{a}}\varphi^{\mathrm{a}}_1 dy. \quad (3.20)$$

The adjoint problem is

$$L^{\mathrm{a}}\varphi^{\mathrm{a}}_1 = 0, \quad (3.21)$$

$$\varphi^{\mathrm{a}}_1(\pm\infty) = 0. \quad (3.22)$$

Integrating the left-hand side of (3.20) by parts and using boundary conditions (3.7), (3.22), we obtain

$$L^{\mathrm{a}}\varphi^{\mathrm{a}}_1 \equiv \varphi^{\mathrm{a}}_{1yy}\left(U - c - SU\frac{i}{k} - B\frac{i}{k}\right) + \varphi^{\mathrm{a}}_{1y}\left(2U_y - SU_y\frac{i}{k} - 2\frac{U}{R}\right)$$

$$+\varphi^{\mathrm{a}}_1\left(k^2c - k^2U + \frac{ik}{2}SU - 2\frac{U_y}{R} + Bik\right). \quad (3.23)$$

Solvability condition for (3.16) has the form

$$\int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \left(\left(c_{\mathrm{g}} - U\right)\varphi_{1yy} - 2\frac{U}{R}\varphi_{1y} + \begin{pmatrix} -2k^2 c + 3k^2 U + U_{yy} \\ -k^2 c_g + ikUS + 2Bik \end{pmatrix} \varphi_1 \right) dy = 0. \quad (3.24)$$

Hence, the group velocity c_{g} can be found from (3.24).

Collecting the terms that contain ε^3 we obtain

$$L_1 \psi_3 = \widetilde{f}_3 \ . \quad (3.25)$$

The evolution equation for the amplitude function $A(\xi,\tau)$ is determined from the solvability condition at the third order. Multiplying the right-hand side of (3.25) by φ_1^a, using (3.13) and the solutions to the boundary value problems (3.14) – (3.19), we obtain the complex Ginzburg-Landau equation for the amplitude $A(\xi,\tau)$ of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A, \quad (3.26)$$

where

$$\sigma = \frac{\sigma_1}{\eta}, \ \delta = \frac{\delta_1}{\eta}, \ \mu = \frac{\mu_1}{\eta} \quad (3.27)$$

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \left(\varphi_{1yy} - k^2 \varphi_1\right) dy, \quad (3.28)$$

$$\sigma_1 = \frac{S}{2} \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \left(-k^2 U \varphi_1 + 2U_y \varphi_{1y} + 2U \varphi_{1yy}\right) dy, \quad (3.29)$$

$$\delta_1 = \int_{-\infty}^{+\infty} \varphi_1^{\mathrm{a}} \begin{pmatrix} \left(c_{\mathrm{g}} - U\right)\varphi_{2yy}^{(1)} - 2\frac{U}{R}\varphi_{2y}^{(1)} \\ + \varphi_2^{(1)} \left(-k^2 c_g - 2k^2 c + 3k^2 U + U_{yy} - ikSU - 2ikB\right) \\ + \varphi_1 \left(2ikc_{\mathrm{g}} + ikc - 3ikU - U\frac{S}{2} - B\right) \end{pmatrix} dy, \quad (3.30)$$

$$\mu_1 = -\int_{-\infty}^{+\infty} \varphi_1^{a} \begin{pmatrix} 6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} + 3ik^3\varphi_1^*\varphi_{2y}^{(2)} \\ + ik^3\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) - ik\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) \\ + ik\varphi_{2y}^{(2)}\varphi_{1yy}^* - ik\varphi_1^*\varphi_{2yyy}^{(2)} \\ + ik\varphi_1\left(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}\right) + 2ik\varphi_{1yyy}^*\varphi_2^{(2)} \\ -\dfrac{S}{2}\begin{pmatrix} -k^2\varphi_1\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 3k^2\varphi_1^*\varphi_{2y}^{(2)} - \dfrac{3k^4}{2u_0}\varphi_1^2\varphi_1^* \\ + 2\varphi_{1yy}\left(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}\right) + 2\varphi_{1yy}^*\varphi_{2y}^{(2)} \\ + 2\varphi_{1y}\left(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}\right) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^* \\ \end{pmatrix} \\ -2\dfrac{ik}{R}\left(\varphi_{2y}^{(2)}\varphi_{1y}^* + \varphi_{2y}^{(0)}\varphi_{1y}\right) \end{pmatrix} dy. \quad (3.31)$$

The coefficients of the Ginzburg-Landau equation (3.26) can be computed using formulas (3.27) – (3.31). Note that in order to perform calculations, it is necessary to solve the linear stability problem (3.6) – (3.7), the corresponding adjoint problem (3.21) – (3.22), three boundary value problems (3.14) – (3.19) and numerically evaluate integrals in (3.27) – (3.31).

4. Spatial Stability of Slightly Curved Shallow Mixing Layers

4.1. Linear Stability

There are two basic approaches to the analysis of linear stability of a base flow in fluid mechanics:

(a) temporal stability analysis;
(b) spatial stability analysis [11].

In both cases, the analysis is performed using the method of normal modes: perturbations are assumed to be proportional to $\exp(i(\alpha x - \beta t))$, where both parameters α and β can be complex: $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$.

In case (a), the wave number $\alpha = \alpha_r$ is real while β is complex. For the case of spatial stability analysis $\beta = \beta_r$ is real and the wave number α is complex: $\alpha = \alpha_r + i\alpha_i$. From a computational point of view, temporal stability analysis is simpler since the corresponding eigenvalue problem is linear with respect to eigenvalue β. On the other hand, a spatial eigenvalue problem is nonlinear in α.

Introducing the stream function (Section 2.1) by the relations (2.1), we can rewrite (1.1) – (1.3) ([12], [19], [20], [21], [26]) in the form (2.4) or

$$L_1\psi_1 = 0, \tag{4.1}$$

where

$$\begin{aligned} L_1\psi &\equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}(\psi_{xxx} + \psi_{xyy}) - \psi_{0yyy}\psi_x \\ &+ \frac{c_f}{2h}(\psi_{0y}\psi_{xx} + 2\psi_{0yy}\psi_y + 2\psi_{0y}\psi_{yy}) + \frac{2}{R}\psi_{0y}\psi_{xy} \end{aligned}. \tag{4.2}$$

Method of normal modes is used to solve (4.2), that is, the perturbation ψ_1 is represented in the form

$$\psi_1(x,y,t) = \varphi(y)\mathrm{e}^{i(\alpha x - \beta t)}, \tag{4.3}$$

where $\varphi(y)$ is the amplitude of the normal perturbation.

Substituting (4.3) into (4.2) and denoted $U = \psi_{0y}$ we obtain the following differential equation

$$\begin{aligned} &\varphi_{yy}\left(\alpha U - \beta - iSU\right) - iSU_y\varphi_y + \frac{2U\alpha}{R}\varphi_y \\ &+ \varphi\left(\alpha^2\beta - \alpha^3 U - \alpha U_{yy} + \frac{i\alpha^2 US}{2}\right) = 0 \end{aligned} \tag{4.4}$$

with the boundary conditions

$$\varphi(\pm\infty) = 0. \tag{4.5}$$

Problem (4.4), (4.5) is an eigenvalue problem. Base flow $U(y)$ is said to be linearly stable if all $\alpha_i > 0$ and unstable if at least one $\alpha_i < 0$. As it is mentioned above, the corresponding problem is linear with respect to β but nonlinear with respect to α. Hence, the following computational procedure is suggested for the solution to the problem. Assuming that both α and β are complex of the form $\alpha = \alpha_r + i\alpha_i$, $\beta = \beta_r + i\beta_i$, for each fixed S, α_r and β_r we calculate α_i such that $\beta_i = 0$. This is achieved by solving linear generalized eigenvalue problem and selecting the new approximation to β_i using a bisection method. Then we change α_r (for the fixed value of S) and repeat the calculation. The region of spatial instability is described by the relation $\alpha_i < 0$.

The base flow is selected in the form

$$U(y) = \frac{1}{2}\left(1 + \tanh y\right) \tag{4.6}$$

The growth rates $-\alpha_i$ versus β_r are shown in Figs. 4.1 and 4.2.

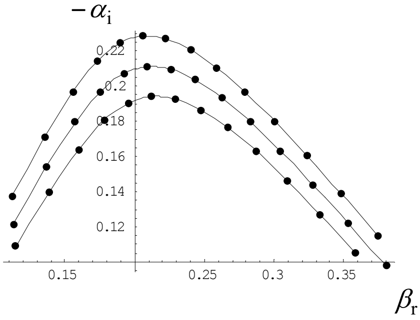


Fig. 4.1. Growth rates $-\alpha_i$ versus β_r for three values of 1/R=0; 0.025; 0.05 (from top to bottom).

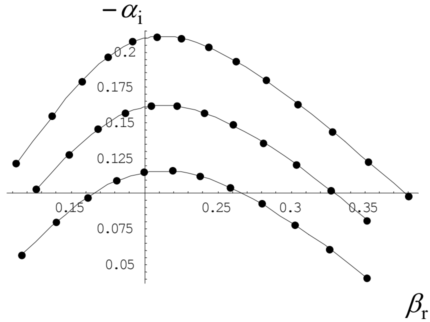


Fig. 4.2. Growth rates $-\alpha_i$ versus β_r for three values of $S = 0; 0.05; 0.1$ (from top to bottom).

The first set of calculations is performed for the case without bottom friction ($S = 0$). It follows from Fig. 4.1 that curvature has a stabilizing influence on the flow (the growth rates decrease as the curvature increases). As can be seen from Fig. 4.2, the increase of the values of S also leads to a more stable flow – the growth rates decrease as the parameter S grows. It is shown that both the bottom friction and flow curvature have a stabilizing influence on the flow.

M. Gaster [31] suggested a transformation, which can be used to approximate spatial growth rates if temporal growth rates are known. However, Gaster's transformation can be used only in the vicinity of the marginal stability curve.

Following M. Gaster [31], we denote by (*T*) and (*Sp*) the solutions to (4.4), (4.5) corresponding to temporal and spatial problems, respectively. It is shown in [31] that near the marginal stability curve:

$$\alpha_r(T) = \alpha_r(Sp),\ \beta_r(T) = \beta_r(Sp),\ \alpha_i(Sp) = -\frac{\beta_i(T)}{c(T)},\ c(T) = \frac{\beta_r(T)}{\alpha_r(T)}.$$

It follows from the Gaster's transformation that on the stability boundary either spatial or temporal stability analyses can be used since in this case $\alpha_i(Sp) = \beta_i(T) = 0$. If the objective of the analysis is to construct a marginal stability curve, then it is recommended to use a temporal stability analysis, which is a simpler method from a computational point of view.

4.2. Weakly Nonlinear Stability

In this section, we describe the second approach that can be used in order to derive an amplitude evolution equation under the assumption that the base flow is not parallel but slightly changes downstream.

Consider the system of shallow water equations of the form (1.1) – (1.3). Let $\psi(x,y,t)$ be the stream function of the flow. Using (2.1) the system of shallow water equations reduces to the following equation for the stream function

$$
\begin{aligned}
&(\Delta\psi)_t + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{2}{R}\psi_y\psi_{xy} + \frac{c_{\mathrm{f}}}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} \\
&+ \frac{c_{\mathrm{f}}}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) + B\Delta\psi = 0.
\end{aligned}
\tag{4.7}
$$

Assume that λ is the wave length of perturbation and l is the length scale of the longitudinal variation of the base flow. In shallow mixing layers (see [59], [60]), the following condition is usually satisfied: $\lambda << l$. Thus, a small parameter ε can be defined as follows: $\varepsilon = \lambda / l$. Following [35] we introduce a slow longitudinal coordinate X by the relation $X = \varepsilon x$. The base flow velocity components are $U(y,X)$ and $\varepsilon V(y,X)$, respectively. The stream function $\psi(x,y,t)$ is represented as the sum of the basic part $\psi_0(y,X)$ and fluctuating part $\psi'(x,y,t)$:

$$\psi(x,y,t) = \psi_0(y,X) + \psi'(x,y,t) \ . \tag{4.8}$$

In addition,

$$U(y,X) = \frac{\partial\psi_0}{\partial y} \ \text{and} \ V(y,X) = -\frac{\partial\psi_0}{\partial X} \ . \tag{4.9}$$

Using (4.8) in (4.7), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + U\frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - \frac{\partial\psi}{\partial x}\frac{\partial^2 U}{\partial y^2} \\
&+ \frac{c_{\mathrm{f}}}{2h}\left(U\left(\frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial^2\psi}{\partial y^2}\right) + 2\frac{\partial U}{\partial y}\frac{\partial\psi}{\partial y}\right) + B\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \\
&+ \varepsilon\begin{pmatrix} \dfrac{\partial^2 U}{\partial y\partial X}\dfrac{\partial\psi}{\partial y} + V\dfrac{\partial}{\partial y}\left(\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2}\right) \\ + \dfrac{c_{\mathrm{f}}}{2h}\left(2\dfrac{\partial U}{\partial X}\dfrac{\partial\psi}{\partial x} - 2V\dfrac{\partial^2\psi}{\partial y\partial x} + \dfrac{U_y}{U}V\dfrac{\partial\psi}{\partial x}\right) \\ + \dfrac{2}{R}\dfrac{\partial U}{\partial X}\dfrac{\partial\psi}{\partial y} \end{pmatrix} = 0.
\end{aligned}
\tag{4.10}
$$

Using the *WKBJ* approximation (see [35]), we represent the stream function in the form

$$\psi(x,y,t)=\varphi(y,X)\exp\left(i\left(\frac{\theta(X)}{\varepsilon}-\omega t\right)\right), \tag{4.11}$$

where $\varphi(y,X)$ – a slow-varying amplitude function;

$\frac{\theta(X)}{\varepsilon}$ – a fast-varying phase function.

The amplitude function $\varphi(y,X)$ is expanded in a power series of the form

$$\varphi(y,X)=\varphi_1(y,X)+\varepsilon\varphi_2(y,X)+... \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), we obtain the following equation at the leading order:

$$L\varphi_1=0, \tag{4.13}$$

where

$$L\varphi_1=\varphi_1''-k^2\varphi_1-\frac{U''}{U-\frac{\omega}{k}}\varphi_1+\frac{2}{R}U\varphi_1'+B\frac{ik}{U-\frac{\omega}{k}}\varphi_1 \\ -\frac{ic_{\mathrm{f}}}{2h(kU-\omega)}\left(-Uk^2\varphi_1+2U\varphi_1''+2U'\varphi_1'\right). \tag{4.14}$$

Here primes denote the derivatives with respect to y and $\theta_x=k$. Using equation (4.12) with boundary conditions $\varphi_1(\pm\infty)=0$ we obtain linear stability problem, where X appears as the parameter. The corresponding eigenfunction of the linear stability problem, $\varphi_1(y,X)$, is represented in the form

$$\varphi_1(y,X)=A(X)\Phi(y,X), \tag{4.15}$$

where $A(X)$ – a slowly varying amplitude;

$\Phi(y,X)$ – a normalized eigenfunction.

At the next order, the following equation is obtained:

$$L\varphi_2=F. \tag{4.16}$$

Equation (4.16) has a solution if and only if the right-hand side F is orthogonal to all eigenfunctions $\widetilde{\Phi}$ of the corresponding adjoint problem:

$$\int_{-\infty}^{+\infty}F\widetilde{\Phi}dy=0. \tag{4.17}$$

Using (4.16) and (4.17), we obtain the following equation for unknown amplitude $A(X)$

$$M(X)\frac{dA}{dX}+N(X)A=0, \tag{4.18}$$

$$M(X)=i\int_{-\infty}^{+\infty}\tilde{\Phi}\begin{pmatrix}2\omega k\Phi-3Uk^2\Phi+U\Phi''-\Phi U'' \\ -2\frac{U}{R}\Phi'+\frac{c_{\mathrm{f}}}{h}iUk\Phi+2ikB\Phi\end{pmatrix}\Big/(kU-\omega)dy \tag{4.19}$$

$$N(X)=i\int_{-\infty}^{+\infty}\frac{\mathrm{D}\tilde{\Phi}}{kU-\omega}dy, \tag{4.20}$$

$$\begin{aligned}\mathrm{D}&=2\omega k\frac{\partial\Phi}{\partial x}+\Phi(\omega-3Uk)\frac{dk}{dX}-\left(3Uk^2+U''\right)\frac{\partial\Phi}{\partial X}+U\frac{\partial\Phi''}{\partial X}\\&+\frac{\partial U'}{\partial X}\Phi'-2\frac{U}{R}\frac{\partial\Phi'}{\partial X}+V\left(\Phi'''-k^2\Phi\right)+B\left(2ik\frac{\partial\Phi}{\partial X}+i\frac{dk}{dX}\Phi\right)\\&+\frac{c_{\mathrm{f}}}{2h}\left(2iUk\frac{\partial\Phi}{\partial x}+2ik\frac{\partial U}{\partial X}\Phi+i\frac{dk}{dX}U\Phi-2ikV\Phi'\right).\end{aligned} \tag{4.21}$$

As a result, the fluctuating part of the stream function has the form

$$\psi(x,y,t)\sim A(X)\Phi(y,X)\times\exp\left(i\left(\frac{1}{\varepsilon}\int_0^X k(X)dX-\omega t\right)\right). \tag{4.22}$$

Formula (4.22) takes into account slow longitudinal variation of the base flow. It is shown in [8] that in a similar asymptotic formula the growth rate and phase speed of perturbation depend not only on the choice of flow quantities but also on the location of the point (x,y), where these quantities are calculated.

Hence, a meaningful comparison of the weakly nonlinear model (4.18) can be made only if a particular quantity of interest Q is selected (for example, longitudinal velocity component or pressure). In this case (see [8]), a local wave number k_{L} can be defined by the formula

$$k_{\mathrm{L}}(x,y)=-i\frac{\partial}{\partial x}\ln Q(x,y), \tag{4.23}$$

where $k_{\mathrm{L}}=k_{\mathrm{Lr}}+ik_{\mathrm{Li}}$.

Thus, in order to compare the weakly nonlinear model (4.18) with experimental data, the following steps should be performed:

- to select a flow quantity Q;
- to measure the quantity Q at some point (x,y);
- to compute the right-hand side of (4.23) at the same point (x,y).

In summary, the weakly nonlinear model (4.18) can be validated if detailed experimental data or numerical results of the solution to nonlinear shallow water equations are available.

5. Stability of Shallow Mixing Layers with Variable Friction

5.1. Linear Stability

Linear stability problem for the case where the friction coefficient is varied in the transverse direction ([17], [18], [22], [28]) is analysed.

The dependence of the friction coefficient $c_{\mathrm{f}}(y)$ on the transverse coordinate y is assumed to be of the form

$$c_{\mathrm{f}}(y) = c_{\mathrm{f}_0}\gamma(y), \tag{5.1}$$

where $\gamma(y)$ is an arbitrary differentiable "shape" function.

The derivative of $c_{\mathrm{f}}(y)$ with respect to y is

$$c_{\mathrm{f}_y}(y) = c_{\mathrm{f}_0}\gamma'(y), \tag{5.2}$$

Shallow water equations (1.1) – (1.3) can be rewritten in the form

$$L_1\psi_1 = 0, \tag{5.3}$$

where

$$\begin{aligned} L_1\psi &\equiv \psi_{xxt} + \psi_{yyt} + \psi_{0y}\psi_{xxx} + \psi_{oy}\psi_{yyx} - \psi_{0yyy}\psi_x \\ &+ \frac{c_{\mathrm{f}}}{2h}\left(\psi_{0y}\psi_{xx} + 2\psi_{oyy}\psi_y + 2\psi_{0y}\psi_{yy}\right) + \frac{c_{\mathrm{f}_y}}{h}\psi_{0y}\psi_y. \end{aligned} \tag{5.4}$$

Using the method of normal modes, the function ψ_1 has the form

$$\psi_1(x,y,t) = \varphi(y)\mathrm{e}^{i(\alpha x - \beta t)}. \tag{5.5}$$

Substituting (5.5) into (5.4), we obtain the boundary value problem

$$\begin{aligned} &\varphi_{yy}\left(\alpha U - \beta - i\gamma SU\right) - i\gamma SU_y\varphi_y - i\gamma' SU\varphi_y \\ &+ \varphi\left(\alpha^2\beta - \alpha^3 U - \alpha U_{yy} + \frac{i\alpha^2 U\gamma S}{2}\right) = 0, \end{aligned} \tag{5.6}$$

$$\varphi(\pm\infty) = 0. \tag{5.7}$$

The following profiles of the base flow velocity $U(y)$ and shape function $\gamma(y)$ are used to compute growth rates of unstable perturbations:

$$\gamma(y) = \frac{1}{2}(1 + \tanh y), \tag{5.8}$$

$$U(y) = \frac{1}{2}(1 - \tanh y). \tag{5.9}$$

The choice of the shape function $\gamma(y)$ in (5.9) is based on the following. First, with a stronger resistance force the base flow velocity becomes smaller. Second, we would like to remove discontinuity in the friction force used in [61] and consider a more realistic case of continuous resistance, which is changing with respect to the transverse coordinate.

Figure 5.1 plots growth rates for the unstable mode for three values of the parameter S : 0.05, 0.10 and 0.15 (from top to bottom). It is seen from Fig. 5.1 that with smaller S the growth rate is larger. In order to see the effect of varying friction more clearly, we plot in Fig. 5.2 growth rates for the most unstable mode for the same three values of S , under the assumption that $\gamma(y) = 1$ (that is, for the case of constant friction coefficient). It is seen from Fig. 5.2 that the increase in S has a stabilizing influence on the flow. However, comparing Figs. 5.1 and 5.2, the overall growth rates for the case of non-uniform friction are larger than for the case of uniform friction. This fact is clearly seen from Fig. 5.3.

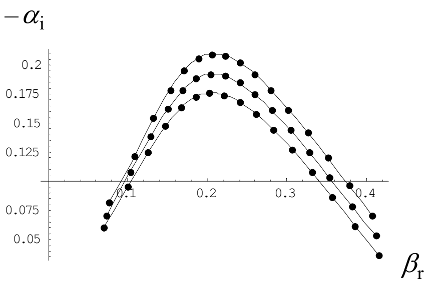


Fig. 5.1. Growth rates $-\alpha_i$ for the shape function given by (5.8).

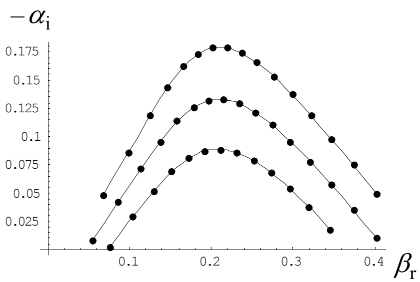


Fig. 5.2. Growth rates $-\alpha_i$ for constant friction coefficient.

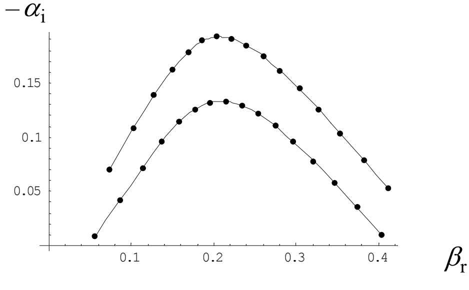


Fig. 5.3. Growth rates $-\alpha_i$ for the case $S = 0.1$
(variable friction – top curve, constant friction – bottom curve).

In the previous example, we considered the case of a symmetric profile; however, experimental data [61] showed that the base flow velocity profile was asymmetric with respect to the transverse coordinate.

Two-parameter profiles ([23], [24]) of the base flow velocity $U(y)$ are used to compute growth rates of unstable perturbations

$$U(y) = \begin{cases} 1 + r \tanh y, & y < 0 \\ 1 + \dfrac{r}{\delta} \tanh \delta y, & y > 0 \end{cases} \tag{5.10}$$

We obtain the following equation (details, for example, in [7]):

$$\varphi''\left(ik(U - c) + \frac{c_{\mathrm{f}}}{h} U \right) + \varphi' S U_y + \varphi \begin{pmatrix} ik^3 c - ik^3 U - ikU_{yy} \\ -\dfrac{k^2 c_{\mathrm{f}} U}{2h} \end{pmatrix} = 0, \tag{5.11}$$

The boundary conditions are

$$\varphi(\pm\infty) = 0. \tag{5.12}$$

Following [61] we assume that the drag force has the form

$$D = \begin{cases} 1/2 \rho (C_D a + c_{\mathrm{f}} / h) U_1^2, & y < 0 \\ 1/2 \rho c_{\mathrm{f}} / h U_2^2, & y > 0 \end{cases} \tag{5.13}$$

where ρ – the density of the fluid;

C_D – the mean drag coefficient;

a – the average solid frontal area per unit volume in the plane perpendicular to the flow [61].

The drag differential between the layer with vegetation and the main channel is described by a dimensionless parameter

$$\gamma = \frac{C_D a}{C_D a + 2 c_{\mathrm{f}} / h} \tag{5.14}$$

In addition, the total resistance can be measured by the generalized bed-friction number

$$S = \left(\frac{C_D a}{2} + \frac{c_{\mathrm{f}}}{h}\right) b, \tag{5.15}$$

where b is the width of the shear layer.

Using (5.11), (5.13), (5.14) and (5.15), we rewrite equation (5.11) in the form

$$(c-U)\varphi_{yy} + (U_{yy} + k^2(U-c))\varphi \ = \frac{SH(y)U}{ik}(\varphi_{yy} + \frac{U_y\varphi_y}{U} - \frac{k^2\varphi}{2}) \quad (5.16)$$

where

$$H(y) = \begin{cases} 1+\gamma, & y<0 \\ 1, & y=0 \\ 1-\gamma, & y>0 \end{cases} \quad (5.17)$$

Problem (5.16), (5.12) is solved numerically by means of a collocation method based on Chebyshev polynomials. Software package *IMSL* is used to solve this problem.

In order to avoid discontinuity at $y=0$ the values of H are replaced by a hyperbolic tangent function of the form $\tanh \delta y$ with large δ values.

In order to compare the results obtained for asymmetric velocity profile (5.10) with the symmetric case, we used the following symmetric velocity profile:

$$U(y) = 1 - \frac{r}{2} + \frac{r}{2\delta} + \left(\frac{r}{2} + \frac{r}{2\delta}\right)\tanh y \quad (5.18)$$

Both profiles (5.10) and (5.18) have the same asymptotes as $y \to \pm\infty$. The graphs of the base flow velocity profiles (5.10) and (5.18) are shown in Figs. 5.4 and 5.5 for two values of δ. The role of the parameter *r* is clearly seen from Figs. 5.4 and 5.5: for smaller values of *r* the horizontal asymptote is reached at larger values of y if the base flow velocity profile is asymmetric with respect to the transverse coordinate y.

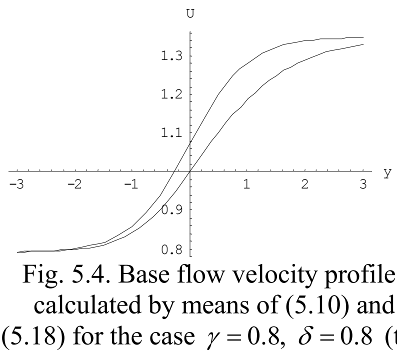


Fig. 5.4. Base flow velocity profiles calculated by means of (5.10) and (5.18) for the case $\gamma = 0.8,\ \delta = 0.8$ (top and bottom curves, respectively).

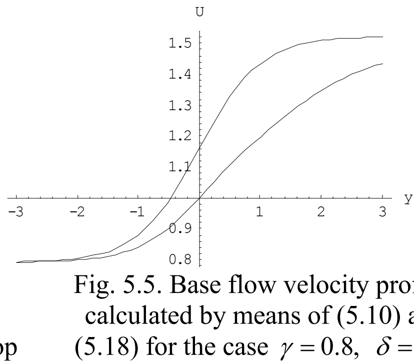


Fig. 5.5. Base flow velocity profiles calculated by means of (5.10) and (5.18) for the case $\gamma = 0.8,\ \delta = 0.6$ (top and bottom curves, respectively).

Stability curves in the (k, S) plane for different values of the parameters of problem (5.16), (5.12) are shown in Figs. 5.6–5.8. Marginal stability curves are shown for the symmetric case (base flow of the form (5.18), solid curve) and asymmetric case (base flow of the form (5.10), dashed curve).

The stabilizing influence of asymmetry of the base flow is also clearly seen in Figs. 5.7 and 5.8. The asymmetric flow becomes more stable since the critical value of the parameter S becomes smaller. In addition, the range of unstable values of k also decreases.

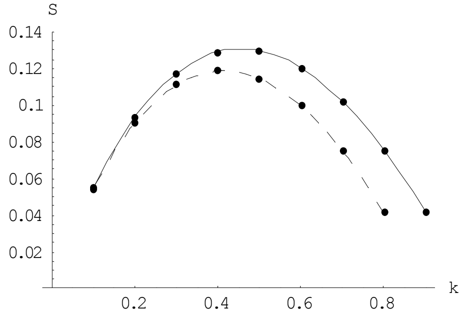


Fig. 5.6. Marginal stability curves for the case $\gamma = 0.8,\ \delta = 0.8$.

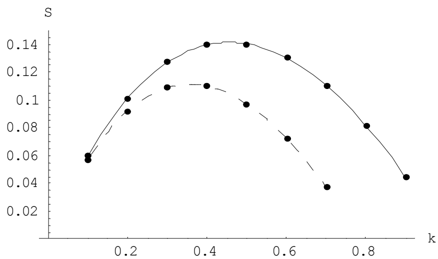


Fig. 5.7. Marginal stability curves $\gamma = 0.8,\ \delta = 0.6$.

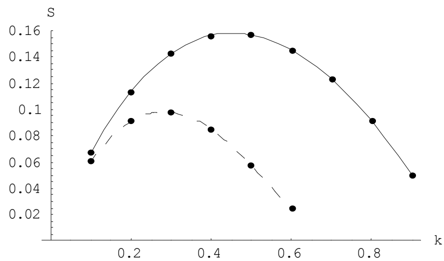


Fig. 5.8. Marginal stability curves $\gamma = 0.8,\ \delta = 0.4$.

Numerical calculations showed stabilizing influence of asymmetry of the base flow profiles: both critical values of the stability parameter and the range of unstable wave numbers decreased when the asymmetry became more pronounced.

5.2. Weakly Nonlinear Stability

Linear stability analysis is a powerful tool that allows setting the conditions under which the flow loses stability. On the other hand, the linear stability analysis does not describe perturbation development. In case the growth rate is relatively small, the perturbation amplitude development equation can be used to compile a weakly nonlinear analysis.

Let us go back to (5.3). Using the method of multiple scales perturbation ψ_1 is sought in the form (see Section 2.1):

$$\psi(x,y,t)=\varphi(y)\mathrm{e}^{ik(x-ct)}\,. \tag{5.19}$$

We obtain:

$$\left(Uk-ck-i\gamma SU\right)\varphi_1''-i\left(\gamma SU_y+\gamma_y SU\right)\varphi_1'+\begin{pmatrix}k^3c-k^3U-kU_{yy}\\ +ik^2\gamma\dfrac{S}{2}U\end{pmatrix}\varphi_1=0 \tag{5.20}$$

$$\varphi(\pm\infty)=0. \tag{5.21}$$

Collecting the terms of order ε^2 we obtain the following equation:

$$L\psi_2=\widehat{f}_2\,. \tag{5.22}$$

Thus, using the function ψ_2 calculated according to (2.22), we obtain three boundary value problems

- for function $\varphi_2^{(0)}$

$$\begin{aligned}&2\gamma S\left(U_y\varphi_{2y}^{(0)}+U\varphi_{2yy}^{(0)}\right)+2\gamma_y SU=ik\begin{pmatrix}\varphi_{1y}\varphi_{1yy}^{*}-\varphi_{1y}^{*}\varphi_{1yy}\\ +\varphi_1\varphi_{1yyy}^{*}-\varphi_1^{*}\varphi_{1yyy}\end{pmatrix}\\&-\frac{\gamma S}{2}\begin{pmatrix}k^2\varphi_1\varphi_{1y}^{*}+k^2\varphi_1^{*}\varphi_{1y}\\ +2\varphi_{1y}\varphi_{1yy}^{*}+2\varphi_{1y}^{*}\varphi_{1yy}\end{pmatrix}-\frac{\gamma_y S}{2}\left(\varphi_1\varphi_1^{*}+2\varphi_{1y}\varphi_{1y}^{*}\right)\end{aligned} \tag{5.23}$$

$$\varphi_2^{(0)}(\pm\infty)=0. \tag{5.24}$$

- for function $\varphi_2^{(1)}$

$$\begin{aligned}&\begin{pmatrix}ikU-ikc\\ +\gamma SU\end{pmatrix}\varphi_{2yy}^{(1)}+S\left(\gamma U_y+\gamma_y U\right)\varphi_{2y}^{(1)}+\begin{pmatrix}ik^3c-ik^3U\\ -ikU_{yy}-\dfrac{\gamma S}{2}Uk^2\end{pmatrix}\varphi_2^{(1)}\\&=\left(c_g-U\right)\varphi_{1yy}+\left(-c_gk^2-2k^2c+3Uk^2+U_{yy}-i\gamma SUk\right)\varphi_1\end{aligned} \tag{5.25}$$

$$\varphi_2^{(1)}(\pm\infty)=0. \tag{5.26}$$

- for function $\varphi_2^{(2)}$

$$\begin{pmatrix} 2ikU \\ -2ikc+\gamma SU \end{pmatrix}\varphi^{(2)}_{2yy}+\begin{pmatrix} \gamma SU_y \\ +\gamma_y SU \end{pmatrix}\varphi^{(2)}_{2y}+\begin{pmatrix} 8ik^3c-8ik^3U \\ -2ikU_{yy}-2\gamma Sk^2U \end{pmatrix}\varphi^{(2)}_{2}$$
$$=ik\left(\varphi_1\varphi_{1yyy}-\varphi_{1y}\varphi_{1yy}\right)+\frac{\gamma S}{2}\begin{pmatrix} 3k^2\varphi_1\varphi_{1y} \\ -2\varphi_{1y}\varphi_{1yy} \end{pmatrix}+\frac{\gamma_y S}{2}\left(\frac{k^2}{2}\varphi_1^2+\varphi_1^2\right) \quad (5.27)$$

$$\varphi^{(2)}_2(\pm\infty)=0. \quad (5.28)$$

Solving three boundary value problems (5.23) – (5.24), (5.25) – (5.26) and (5.27) – (5.28) numerically we obtain the functions $\varphi^{(0)}_2(y), \varphi^{(1)}_2(y)$ and $\varphi^{(2)}_2(y)$. The function ψ_2 (the second order correction) is then given by (2.22).

The adjoint boundary value problem is in the form

$$L^{\mathrm{a}}\varphi^{\mathrm{a}}_1=0 \quad (5.29)$$

$$\varphi^{\mathrm{a}}_1(\pm\infty)=0, \quad (5.30)$$

where

$$L^{\mathrm{a}}\varphi^{\mathrm{a}}_1\equiv\varphi^{\mathrm{a}}_{1yy}\left(U-c-\gamma SU\frac{i}{k}\right)+\varphi^{\mathrm{a}}_{1y}\left(2U_y-\frac{i}{k}\left(\gamma SU_y+\gamma_y SU\right)\right)$$
$$+\varphi^{\mathrm{a}}_1\left(k^2c-k^2U+\frac{ik}{2}\gamma SU\right). \quad (5.31)$$

Applying the solvability condition to (5.29), we obtain (5.32), from which it is possible to find the group velocity c_g.

$$\int_{-\infty}^{+\infty}\varphi^{\mathrm{a}}_1\left((c_g-U)\varphi_{1yy}+(-k^2c_g-2k^2c+3k^2U+U_{yy}-ikU\gamma S)\varphi_1\right)dy=0. \quad (5.32)$$

The third order correction (for ε^3)

$$L\psi_3=f_3. \quad (5.33)$$

Applying the solvability condition to (5.33), we obtain

$$\int_{-\infty}^{\infty}\varphi^{\mathrm{a}}_1 L\psi_3 dy=0. \quad (5.34)$$

Equation (5.34) converted to the amplitude evolution equation for slowly varying amplitude function $A(\xi,\tau)$ of the form

$$\frac{\partial A}{\partial \tau}=\sigma A+\delta\frac{\partial^2 A}{\partial \xi^2}-\mu\,|A|^2\ A. \tag{5.35}$$

Equation (5.35) is the complex Ginzburg-Landau equation with complex coefficients σ,δ and μ

$$\sigma=\frac{\sigma_1}{\eta},\ \delta=\frac{\delta_1}{\eta},\ \mu=\frac{\mu_1}{\eta}. \tag{5.36}$$

where

$$\eta=\int_{-\infty}^{+\infty}\varphi_1^{a}(\varphi_{1yy}-k^2\varphi_1)dy\,, \tag{5.37}$$

$$\sigma_1=\int_{-\infty}^{+\infty}\varphi_1^{a}\left(\frac{\gamma S}{2}\left(-k^2U\varphi_1+2U_y\varphi_{1y}+2U\varphi_{1yy}\right)-\gamma_y SU\varphi_{1y}\right)dy, \tag{5.38}$$

$$\delta_1=\int_{-\infty}^{+\infty}\varphi_1^{a}\begin{pmatrix}\left(c_g-U\right)\varphi_{2yy}^{(1)}\\ +\varphi_2^{(1)}\left(-k^2c_g-2k^2c+3k^2U+U_{yy}-ik\gamma SU\right)\\ +\varphi_1\left(2ikc_g+ikc-3ikU-U\dfrac{\gamma S}{2}\right)\end{pmatrix}dy, \tag{5.39}$$

$$\mu_1=-\int_{-\infty}^{+\infty}\varphi_1^{a}\begin{pmatrix}6ik^3\varphi_2^{(2)}\overset{*}{\varphi}_{1y}-2ik\overset{*}{\varphi}_{1y}\varphi_{2yy}^{(2)}+3ik^3\overset{*}{\varphi}_1\varphi_{2y}^{(2)}\\ +ik^3\varphi_1(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})-ik\varphi_{1yy}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})\\ +ik\varphi_{2y}^{(2)}\overset{*}{\varphi}_{1yy}-ik\overset{*}{\varphi}_1\varphi_{2yyy}^{(2)}\\ +ik\varphi_1(\varphi_{2yyy}^{(0)}+\varphi_{2yyy}^{*(0)})+2ik\overset{*}{\varphi}_{1yyy}\varphi_2^{(2)}\\ -\dfrac{\gamma S}{2}\begin{pmatrix}-k^2\varphi_1(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})+3k^2\overset{*}{\varphi}_1\varphi_{2y}^{(2)}\\ -\dfrac{3k^4}{2u_0}\varphi_1^2\overset{*}{\varphi}_1+2\varphi_{1yy}(\varphi_{2y}^{(0)}+\varphi_{2y}^{*(0)})\\ +2\overset{*}{\varphi}_{1yy}\varphi_{2y}^{(2)}+2\varphi_{1y}(\varphi_{2yy}^{(0)}+\varphi_{2yy}^{*(0)})+2\varphi_{2yy}^{(2)}\overset{*}{\varphi}_{1y}\end{pmatrix}\\ -\gamma_y S\left(-k\varphi_1\varphi_2^{(2)}+\varphi_{1y}\varphi_{2y}^{(2)}\right)\end{pmatrix}dy \tag{5.40}$$

Especially important role in this case is played by the sign of the real part of μ (known as the Landau constant in the literature). The Landau constant had the "wrong sign" in [57], which meant that finite amplitude saturation was not possible and higher order terms (with respect to A) quickly became important so that (5.35) could be used for a very short time period (in other words, practical application of

(5.35) is very limited). In contrast to [57] it is shown in [29], [33] and [43] that for shallow water flows the Landau constant in (5.35) has the “right sign” ($\mu_r > 0$) so that (5.35) can be used. Ginzburg-Landau equation has a rich variety of solutions depending on the values of the coefficients [1].

6. Numerical Solution

It is clear from the discussion in Chapter 2 (Fig. 2.4) that the weakly nonlinear approach can be used in a small neighbourhood of the critical point. Thus, we can apply the theory and compute the coefficients of the Ginzburg-Landau equation. To ensure that the model can adequately represent the dynamics of a fully nonlinear model at least at the initial stage of transition period when the base flow becomes linearly unstable, the paper by Suslov and Paolocci [58] can be used, where a relatively simple criterion is proposed. If the growth rates of unstable perturbation can be well approximated by a parabola in the whole range of unstable wave numbers, then the Ginzburg-Landau equation produces reliable results. In order to test this assertion, we computed growth rates for the range of unstable wave numbers for the following values of the parameters of the problem of stability of slightly curved shallow mixing layers for base flow profile (2.8). The results of calculations are shown in Fig. 6.1 ($S < S_c$) for c_i ($c = c_r + ic_i$, when $c_i > 0$). As can be seen from the figures, the curve representing growth rates and parabolic fit are almost indistinguishable. Thus, we conclude that the Ginzburg-Landau equation can be successfully used to analyse the dynamics of the flow above the threshold.

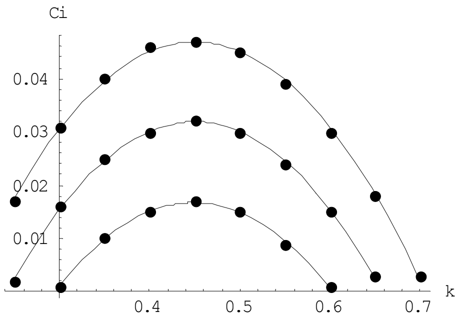


Fig. 6.1. Quadratic approximations of the growth rates for the following values of the parameters *S*=0.09, 0.08, 0.07 and 1/*R*=0.03 (from top to bottom).

Table 6.1 presents the results for the coefficients of the Ginzburg-Landau equation (2.38) using formulas (2.39) – (2.42) of numerical calculations. The results are shown for the base flow profile (2.8) (Fig. 2.1) for the values of 1/R in the range from 0 to 0.04.

Table 6.1. Linear and Weakly Nonlinear Stability Characteristics for Different Values of 1/R (Chapter 2) and Base Flow Profile (2.8)

1/R	0.00	0.01	0.02	0.04
k_c	0.456	0.453	0.449	0.440
S_c	0.065	0.058	0.054	0.047
c_c	1.954	1.965	1.977	2.004
σ	$0.184 - 0.016i$	$0.173 - 0.015i$	$0.163 - 0.013i$	$0.141 - 0.009i$
μ	$2.861 + 0.494i$	$3.046 + 0.539i$	$3.244 + 0.590i$	$3.673 + 0.720i$
c_g	1.927	1.924	1.922	1.914
δ	$6.487+13.238i$	$6.014+13.757i$	$5.472+14.447i$	$4.124+16.524i$

We also present here the calculations in a weakly nonlinear regime for the case of the problem considered in Chapter 5 (the case of non-uniform friction). Base flow and the "shape" profile $\gamma(y)$ are used to model non-uniform friction

$$U(y) = 2 + \tanh y\,,$$

$$\gamma(y) = \frac{\beta+1}{2} + \frac{\beta-1}{2}\tanh(\lambda y)\,.$$

The results of the numerical computations of the linear stability characteristics and the coefficients of the Ginzburg-Landau equation are shown in Table 6.2 below.

Table 6.2. Linear and Weakly Nonlinear Calculations for $\beta = 0.3$

λ	0.25	0.5	1.0	1.5
k_c	0.442	0.437	0.438	0.437
S_c	0.198	0.205	0.211	0.214
c_c	1.972	1.985	2.004	2.018
σ	$0.195 - 0.487i$	$0.195 - 0.080i$	$0.183 - 0.133i$	$0.174 - 0.173i$
μ	$2.374 + 0.690i$	$2.151 + 0.687i$	$2.090 + 0.516i$	$2.092 + 0.262i$
c_g	1.956	1.981	2.007	2.018
δ	$7.077+13.243i$	$7.330 +12.434i$	$7.403+10.752i$	$7.255 + 9.645i$

After rescaling [1], equation (5.35) for the complex amplitude $\widetilde{A}$ has the form:

$$\frac{\partial \widetilde{A}}{\partial \widetilde{\tau}} = \widetilde{A} + (1 + c_1 i)\frac{\partial^2 \widetilde{A}}{\partial \widetilde{\xi}^2} - (1 + c_2 i)|\widetilde{A}|^2\,\widetilde{A}\,, \qquad (6.1)$$

where

$$\widetilde{\tau} = \tau\sigma_{\mathrm{r}}\,, \qquad \widetilde{\xi} = \xi\sqrt{\frac{\sigma_{\mathrm{r}}}{\delta_{\mathrm{r}}}}\,, \qquad \widetilde{A} = A\sqrt{\frac{\mu_{\mathrm{r}}}{\sigma_{\mathrm{r}}}}\mathrm{e}^{-ic_0\sigma_{\mathrm{r}}\tau}\,,$$

$$c_0 = \frac{\sigma_\mathrm{i}}{\sigma_\mathrm{r}}, \qquad c_1 = \frac{\delta_\mathrm{i}}{\delta_\mathrm{r}}, \qquad c_2 = \frac{\mu_\mathrm{i}}{\mu_\mathrm{r}}.$$

Some closed form solutions of (6.1) are known in the literature [1], [10]. One of the simplest solutions is the solution of the form

$$\widetilde{A} = a_0 \mathrm{e}^{iq\widetilde{\xi} + i\omega\widetilde{\tau}}, \tag{6.2}$$

where

$$a_0 = \sqrt{1-q^2}, \qquad \omega = -c_2 - (c_1 - c_2)q^2.$$

Stability of (6.2) can be investigated by assuming that [37]

$$\widetilde{A} = (a_0 + \hat{a}\mathrm{e}^{ik\widetilde{\xi} + \lambda\widetilde{\tau}} + \hat{a}^* \mathrm{e}^{-ik\widetilde{\xi} + \lambda\widetilde{\tau}})\mathrm{e}^{iq\widetilde{\xi} + i\omega\widetilde{\tau}}. \tag{6.3}$$

Substituting (6.3) into (6.1), we obtain equation for λ. For the case of small k the stability condition has the form:

$$1 + c_1 c_2 > 0 \tag{6.4}$$

provided that q satisfies the inequality

$$q^2 < \frac{1 + c_1 c_2}{2c_2^2}. \tag{6.5}$$

Condition (6.4) is known as the Benjamin-Feir stability condition. If (6.4) is not satisfied, then plane wave solutions of (6.2) are unstable (and, therefore, cannot be observed in experiments).

Numerical solutions of the Ginzburg-Landau equation (6.1) are presented below for different values of the parameters c_1, c_2 and different initial conditions. The problem is formulated as follows: to find the solution of (6.1) for the given boundary conditions

$$\widetilde{A}|_{\widetilde{\xi}=0} = 0, \ \widetilde{A}|_{\widetilde{\xi}=L} = 0, \tag{6.6}$$

and the initial condition

$$\widetilde{A}|_{\widetilde{\tau}=0} = f(\widetilde{\xi}). \tag{6.7}$$

Method of lines implemented in Mathematica 5 is used for the numerical solution to problem (6.1), (6.6), (6.7).

Table 6.3 shows numerical values of the coefficients c_1 and c_2 for different values of λ. As can be seen from Table 6.3, condition (6.4) is satisfied for all cases considered.

Table 6.3. Numerical Values of the Coefficients c_1 and c_2 of the Ginzburg-Landau Equation for $\beta = 0.3$

β	0.3000	0.3000	0.3000	0.3000
λ	0.25	0.5	1.0	1.5
$c_1 = \delta_i / \delta_r$	1.8713	1.6963	1.4524	1.3294
$c_2 = \mu_i / \mu_r$	0.2906	0.3194	0.2469	0.1252
$1 + c_1 c_2$	2.1619	2.0157	1.6993	1.4547

The first computation is performed for the case $c_1 = 1.3293$ and $c_2 = 0.1251$. The values of these parameters are taken from Table 6.3. The function in (6.7) is assumed to be small random noise of order 0.01. The results are shown in Fig. 6.3. Since the parameters of the problem satisfy (6.4) and (6.5) (are in the region of stability), the modulus of the amplitude reaches a constant value.

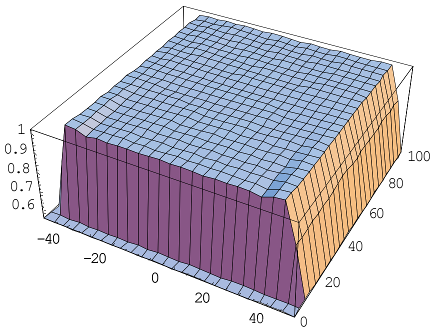


Fig. 6.3. Plot of the $|\tilde{A}|$.

The second set of computations is performed for the case $f(\tilde{\xi}) = \sqrt{1-q^2}\, e^{iq\tilde{\xi}}$ where $q = 0.5$. The results are shown in Fig. 6.4.

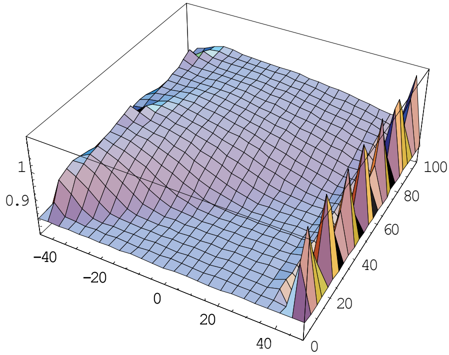


Fig. 6.4. Plot of the $\left|\widetilde{A}\right|$.

Finally, we consider the case, where the Benjamin-Feir stability condition (6.4) is not satisfied. The values of the parameters are taken from [43]: $c_1 = -0.799564$ and $c_2 = 2.189654$ (these parameters correspond to the weakly nonlinear analysis of wake flows). Random noise of order 0.01 is used as the initial condition. The results are shown in Figs. 6.5 and 6.6. As can be seen from Figs. 6.5 and 6.6, stabilization of the amplitude does not occur in this case.

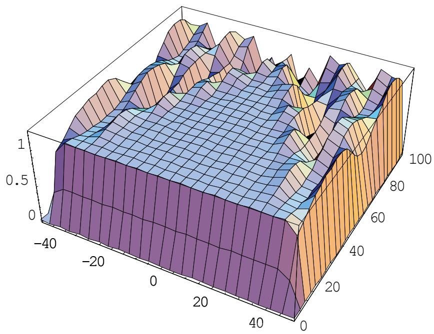


Fig. 6.5. Plot of the $\left|\widetilde{A}\right|$.

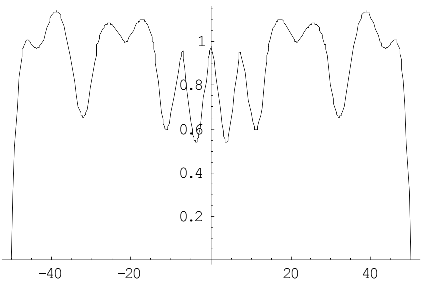


Fig. 6.6. The final configuration of $\left|\widetilde{A}\right|$.

These examples illustrate the well-known fact that the Ginzburg-Landau model is quite rich in terms of different solutions. Illustrative computations in Figs. 6.3–6.6 show that both initial conditions and the values of the coefficients are responsible for spatio-temporal dynamics of the amplitude. The domain of applicability of the Ginzburg-Landau equation has to be defined. The equation is derived as a small neighbourhood of the critical point. Comparison of fully nonlinear simulations with predictions based on the Ginzburg-Landau model is required in order to test the validity of the model. This is left for future research.

Conclusion

The main conclusions from the linear stability analysis are as follows:

- Flow curvature effect is twofold: calculations show that the curvature gives a destabilizing effect on the unstable curved mixing layer and stabilizing effect on the stable curved mixing layer.
- Particle loading parameter has a stabilizing influence on the flow.
- Spatial stability analysis has been performed in the Thesis as well. One of the objectives has been to estimate the accuracy of Gaster's transformation away from the marginal stability curve.
- It is shown that the base flow asymmetry has a stabilizing influence on the flow.
- Calculations show that growth rates for the case of non-constant friction are higher than growth rates for the case of uniform friction.

Two methods of weakly nonlinear theory have been used in the Thesis for the stability analysis of shallow mixing layers. The first method uses parallel flow assumption. Using the method of multiple scales, the complex Ginzburg-Landau equation is derived from shallow water equations for slightly curved shallow water flow mixing layers, for two-component slightly curved mixing layers, for mixing layers with non-uniform friction. The coefficients of the equation are expressed in terms of integrals containing linearized characteristics of the flow.

The second method is based on the assumption that the wave length of perturbation is much smaller than the length scale of longitudinal evolution of the base flow. Perturbed stream function at the leading order is decomposed in this case into a slow-varying amplitude function and a fast-varying phase function. Solvability condition at the second order gives amplitude equation for the unknown amplitude of the most unstable mode.

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