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Using Fuzzy Probability Weights in Cumulative Prospect Theory

Oļegs Užga-Rebrovs[1], Gaļina Kuļešova[2]
[1] *Rezekne Academy of Technologies*, [2] *Riga Technical University*

Abstract – **During the past years, a rapid growth has been seen in the descriptive approaches to decision choice. As opposed to normative expected utility theory, these approaches are based on the subjective perception of probabilities by the individuals, which takes place in real situations of risky choice. The modelling of this kind of perceptions is made on the basis of probability weighting functions. In cumulative prospect theory, which is the focus of this paper, decision prospect outcome weights are calculated using the obtained probability weights. If the value functions are constructed in the sets of positive and negative outcomes, then, based on the outcome value evaluations and outcome decision weights, generalised evaluations of prospect value are calculated, which are the basis for choosing an optimal prospect.**

In cumulative prospect theory, all relevant evaluations are represented in deterministic form. The present research is an attempt to extend classical prospect theory to the cases when the weights of probabilities are given in a fuzzy form.

Keywords – **Fuzzy probability weight, probability weighting function, prospect theory, utility theory.**

I. Introduction

This paper uses the terminology represented in the literature on descriptive theories of choice that differs from the traditional terminology used in the literature on normative approaches to a decision choice. Thus, instead of the concept "decision", a more general concept "prospect" is used. A prospect can be treated as an alternative decision, lottery or other situation of risky choice.

The present research makes use of the concept of value function introduced in the original work on the cumulative prospect theory [1]. Actually, the concept of value function in this context coincides with the concept of utility in expected utility theory except for the conceptual notion of reference point, which is beyond the topic of this paper.

Just after the expected utility theory has been developed [2], it has occupied the leading position as a basis for a decision choice under risk. The theory is normative, i.e., it prescribes how decisions have to be made. It assumes that decision makers are rational individuals, and the maximisation of the expected utility is the basis for decision choice.

Soon after the theory of expected utility has appeared, attempts to study its descriptive properties have started. The attempts have been aimed at discovering how adequately the theory models real behaviour of the individuals in risky situations of choice. Even the first results obtained [3], [4] have shown that the individuals in their actual choices are focused not only on the maximisation of the expected utility but also on the probability of prospect outcomes. These findings contradict the canons of expected utility theory, which states that the probabilities of outcomes are only used as linear coefficients when calculating the expected utility.

A serious blow to the expected utility theory was delivered by Allais paradox [5]. This paradox includes specially constructed situations of choice (lotteries). The results of actual choices on these lotteries have explicitly evidenced that the individuals in their choice largely orient themselves towards the probabilities of lottery outcomes.

In [6], the results of actual choices in risky situations are modelled with triangular diagrams. This paper strictly proves that choices in Allais lotteries are not consistent with normative requirements of expected utility theory.

Visual examples in research [7] clearly show that the Allais paradox relates to two effects: common relation effect and common consequence effect. The results of empirical studies presented in [8] also demonstrate that the results of actual choices of the individuals in risky situations sufficiently differ from those obtained on the basis of expected utility theory.

Since in risky situations of choice the individuals account for the probabilities of prospect outcome occurrence, one can say that they weight these probabilities, on whose basis subjective weighting of outcomes is made. In [9], the following explanation of that phenomenon is given: "Intuitive explanation of the Allais paradox is the fair of disappointment. The loss of guaranteed 5 million dollars in a gamble seems to be much less disappointing than the loss of 5 million dollars with slight chances of occurrence. Due to that, if in a gamble people fare to become disappointed, it is assumed that their preferences really depend on the probabilities".

The presence of multiple empiric evidences about the influence of outcome probabilities on the results of prospect choice has proven indisputably that the expected utility theory cannot cope in principle with the current state of things [10], [11]. The necessity of new descriptive theories of choice has become evident. Rank-dependent utility theory [12] and prospect theory [13], [1] have become widespread.

In this paper, we focus on cumulative prospect theory [1] due to these reasons: (1) currently, this theory is the most developed descriptive theory of prospect choice and (2) the theory can be considered a specific extension of rank-dependent utility theory.

II. General Principles of Weighting Probabilities

The results of numerous empiric studies have shown that in most cases the individuals subjectively overweight probabilities of the outcomes with large wins and subjectively underweight probabilities of the outcomes with large losses. In the area of average values of wins and losses, subjective weights of relevant probabilities nearly correspond to the values of those probabilities.

How could subjective weights of real values of the probabilities be modelled? This can be done through constructing a subjective function for weighting probabilities (weighting function) for an individual in a specific task of choosing prospects. Such a function can be constructed simultaneously with constructing the value function [14].

Quite frequently, approximations of original weighting functions with the help of suitable degree functions are used. The Prelec-I, II [15] and Tversky-Kahneman weighting functions are widespread [1]:

$$w(p)=\frac{p^y}{(p^y+(1-p)^y)^{\frac{1}{y}}},\ 0.292<y<1. \qquad (1)$$

For weighting probabilities in cumulative prospect theory, the authors have proposed these versions of the weighting function (1):

- for positive values of the evaluations of outcomes of prospects:

$$w^+(p)=\frac{p^\gamma}{\left(p^\gamma+(1-p)^\gamma\right)^{\frac{1}{\gamma}}},\ \gamma=0.61. \qquad (2,\ a)$$

- for negative values of the evaluations of outcomes of prospects:

$$w^-(p)=\frac{p^\sigma}{\left(p^\sigma+(1-p)^\sigma\right)^{\frac{1}{\sigma}}},\ \sigma=0.69. \qquad (2,\ b)$$

The graph of weighting function (2, a) is shown in Fig. 1. The graph of weighting function (2, b) just slightly differs from that graph.

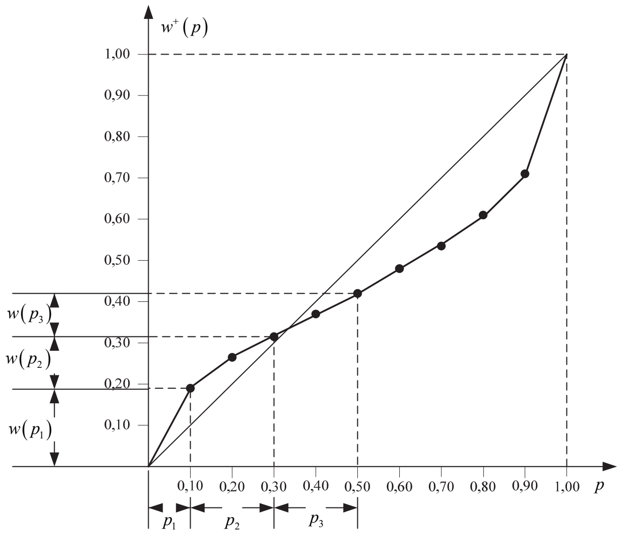


Fig. 1. Graph of weighting function (2, a).

III. Calculation of the Generalised Evaluations of Preference in Cumulative Prospect Theory

Let there be set a prospect $A=\left(k_1,p_1;...;k_n,p_n\right)$, whose outcomes are ordered in the order of decrease of their criteria evaluations:

$$k_1\ge k_2\ge ...\ge k_r\ge 0\ge k_{r+1}\ge k_{r+2}\ge ...\ge k_n \qquad (3)$$

As opposed to the rank-dependent utility theory, this theory allows for negative values of outcome evaluations.

Using certain weights of probabilities $w^+\left(p_i\right)$, $w^-\left(p_j\right)$ and values of outcome probabilities, the values of decision weights of outcomes are calculated by these expressions:

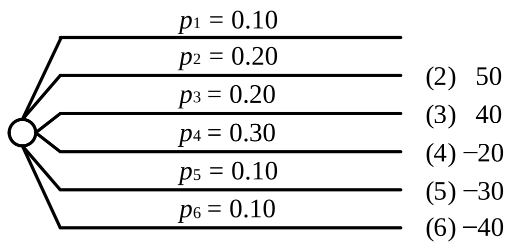
for $i\le r$

$$\pi_i^+=w^+\left(p_i+...+p_1\right)-w^+\left(p_{i-1}+...+p_1\right),$$
$$\pi_1^+=w^+\left(p_1\right); \qquad (4)$$

for $j>r$ $\quad \pi_j^-=w^-\left(p_j+...+p_n\right)-w^-\left(p_{j+1}+...+p_n\right),$

$$\pi_n^-=w^-\left(p_n\right) \qquad (5)$$

Let us consider the method of calculating decision weights using an example. Let us set this prospect:



Probabilities p_1, p_2, and p_3, and the corresponding weights $w\left(p_1\right)$, $w\left(p_2\right)$ and $w\left(p_3\right)$ are shown in Fig. 1.

Decision weights of outcomes of prospect A are calculated by expressions (4) and (5). Let us describe calculations for positive outcomes of the prospects.

$$\pi_1^+=w^+\left(p_1\right)=w^+\left(0.10\right)=0.185;$$
$$\pi_2^+=w^+\left(p_1+p_2\right)-w^+\left(p_1\right)=$$
$$=w^+\left(0.10+0.20\right)-w^-\left(0.10\right)=$$
$$=w^+\left(0.30\right)-w^+\left(0.10\right)=$$
$$=0.318-0.185=0.133;$$
$$\pi_3^+=w^+\left(p_1+p_2+p_3\right)-w^+\left(p_1+p_2\right)=$$
$$=w^+\left(0.10+0.20+0.20\right)-w^+\left(0.10+0.20\right)=$$
$$=w^+\left(0.50\right)-w^+\left(0.30\right)=$$
$$=0.421-0.318=0.103.$$

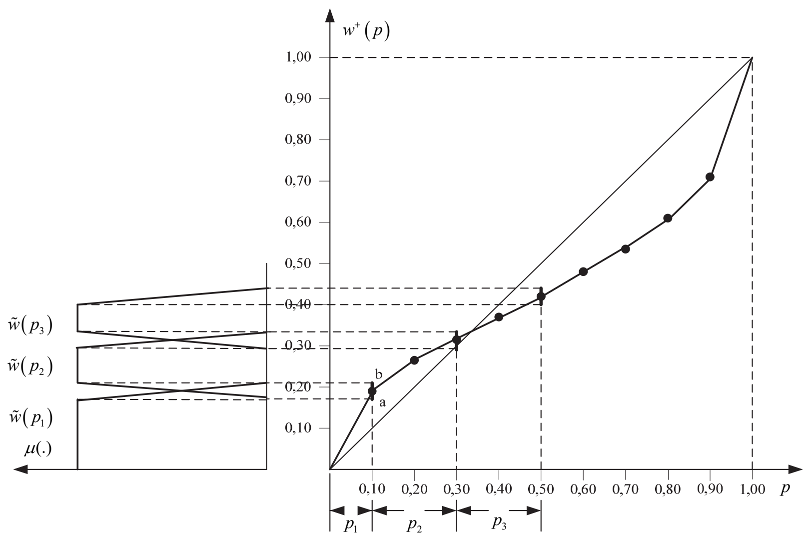


Fig. 2. Modelling uncertainties regarding probability weights.

Let us suppose that in the set of consequences of the available prospects, value functions for positive and negative outcomes are constructed. Using these functions, criteria evaluations of outcomes k_i, $i = 1,...,6$, of prospect A are transformed to the value evaluations $v(k_i)$, $i = 1,...,6$.

Common value of the prospect A is calculated by expression

$$V_{PT}(A) = \sum_{i=1}^{r} v(k_i)\pi_i^+ + \sum_{j=r+1}^{n} v(k_j)\pi_j^-, \qquad (6)$$

where π_i^+ – decision weight of the i-th outcome with positive criteria evaluation calculated by expression (4);
π_j^- – decision weight of the j-th outcome with negative criteria evaluation calculated by expression (5);
$v(k_i)$, $v(k_j)$ – value evaluations of the i-th and j-th outcomes.
If for all prospects A_j, $j = 1,...,m$, their generalised value evaluations
$V_{PT}(A_j)$
are calculated; a prospect having a maximum value $\max_j V_{PT}(A_j)$ is chosen as optimal.

IV. Modelling Uncertainties When Assigning Probability Weights

Probability weighting functions are constructed on the basis of subjective perceptions of outcome probabilities in a specific task of prospect choice. Assumptions that these subjective evaluations are real representations of the status quo and can be represented in the form of deterministic evaluations are invalid simplification of the real state of things. Subjective perceptions of the individuals are in essence vague and uncertain. Distinguishing and fixation of a single weight value for a specific value of probability out of a set of its possible values are of rather deliberate character. It seems necessary to account for and model inevitable uncertainties of the individual's judgements about the weights of probabilities and expand them to further calculations of the weights of outcome decisions and generalised evaluations of the values of prospects.

Common idea of modelling uncertainties regarding probability weights is as follows (see Fig. 2). This figure is based on the graph of the weighting function $w^+(p)$ from Fig. 1 and probabilities as well as their weights shown in Fig. 1. Instead of defining point values of the upper boundaries of the probability weights, it is assumed now that these boundaries have uncertain character. For example, for probability p_1 a minimum reliable value of the upper boundary of its weight $w(p_1)$ (point *a* in Fig. 2) is determined.

Then interval [a,b] is defined within which a real value of the upper boundary of weight $w(p_1)$ can be located. This interval at the same time represents uncertainty regarding the lower boundary of weight $w(p_2)$. Just in the same manner, uncertainties regarding the boundaries between other probability weights are modelled. As a result, we obtain a set of trapezoidal fuzzy values of probability weights. Assuming that the intervals between the boundaries of successive weights of probabilities are symmetric and equal to 0.040, we have

$$\tilde{w}^+(p_1) = (0, 0, 0.165, 0.205)$$

$$\tilde{w}^+(p_2) = (0.165, 0.205, 0.401, 0.441)$$

$$\tilde{w}^+(p_3) = (0.298, 0.338, 0.401, 0.441).$$

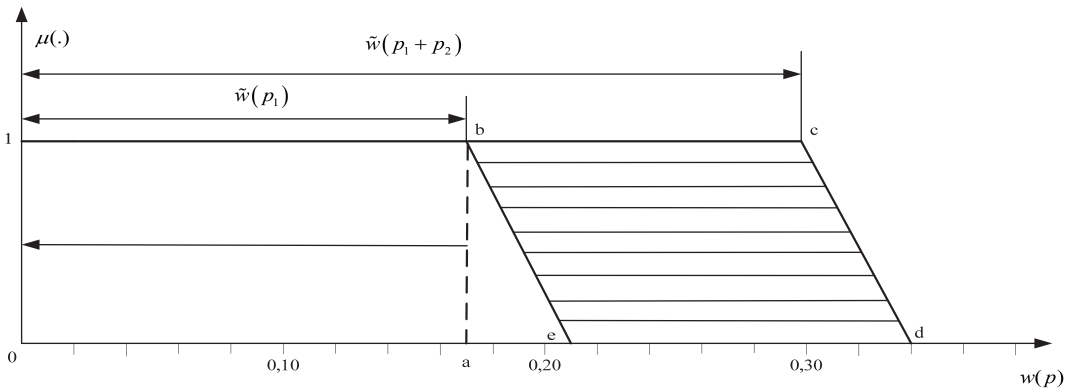


Fig. 3. Schematic representation of the technique of determination of fuzzy decision weight $\tilde{\pi}_2^+$.

With regard to fuzzy values of the weights of summary values of the probabilities that appear in further calculations, they are determined as follows:

$$\tilde{w}^+(p_1+p_2)=(0,0,0.298,0.338)$$

$$\tilde{w}^+(p_1+p_2+p_3)=(0,0,0.401,0.441).$$

These expressions do not account for internal fuzzy boundaries between the addend probabilities. They have no meaning for the resulting fuzzy weights of probabilities. Only the upper fuzzy boundary of the probability weight that appears last in the corresponding sum has meaning.

Let us calculate fuzzy weights of outcome decisions (1), (2) and (3) of the aforementioned prospect:

$$\pi_1^+=w^+(p_1)=(0,0,0.165,0.205).$$

There is no need to make formal calculations of the successive weights of decisions following the rules of fuzzy arithmetic. Let us have a look at Fig. 3. This figure shows graphs of membership functions of fuzzy weights of probabilities $\tilde{w}(p_1)$ and $\tilde{w}(p_1+p_2)$. The shaded area *bcde* schematically represents the difference between those fuzzy weights. Point a corresponds to the minimum reliable value of the upper boundary of weight $\tilde{w}(p_1)$. Interval *ae* expresses uncertainty regarding the real value of that upper boundary. Let us orient ourselves towards the maximum uncertainty of the fuzzy weight of decision $\tilde{\pi}_2^+$. It is evident that trapezoidal fuzzy number *(a, b, c, d)* in Fig. 3 satisfies that uncertainty. To determine a real fuzzy evaluation of $\tilde{\pi}_2^+$, let us move the fuzzy number to the origin. As a result, we have:

$$\tilde{\pi}_2^+=(0,0,0.113,0.153).$$

Following that schema, let us determine fuzzy decision weight $\tilde{\pi}_3^+$: $\tilde{\pi}_3^+=(0,0,0.083,0.123)$. Fuzzy decision weights for negative decision weights of prospect A can be determined in the same way using the weighting function (2, b).

Once the values of fuzzy weights of decisions for all outcomes of prospect A are available, its fuzzy generalised value is calculated as follows:

$$\tilde{V}_{PT}(A)=\sum_{i=1}^{r}v(k_i)\tilde{\pi}_i^+ + \sum_{j=r+1}^{n}v(k_j)\tilde{\pi}_j^- . \quad (7)$$

Expression (7) is a fuzzy version of expression (6).

To choose an optimal prospect, fuzzy numbers $\tilde{V}_{PT}(A_j)$, $j=1,...,m$ have to be compared. There are many methods for comparing fuzzy numbers. However, it is evident that in the case under consideration fuzzy evaluations $\tilde{V}_{PT}(A_j)$, $j=1,...,m$ have the shape of single-sided trapezoidal fuzzy numbers. Due to that, their comparison reduces to a simple comparison of the location of the upper fuzzy boundaries.

V. Conclusion

Constructing probability weighting functions is based on subjective perceptions of some or other probability values of prospect outcome occurrence by the individual. Like any other subjective judgements and evaluations, the values of probability weights are related to considerable uncertainties. It seems necessary to model these prior uncertainties and take them into consideration in further calculations.

One suitable technique to model such uncertainties is to use fuzzy numbers. This paper offers to employ fuzzy boundaries between successive weights of the corresponding probabilities. By specifying intervals of possible values of boundaries between the probability weights, we translate that uncertainty to the fuzzy form of the corresponding boundaries. As a result, fuzzy values of probability weights are obtained in the form of trapezoidal fuzzy numbers. Using the obtained fuzzy values of probability weights, one can sufficiently easily find fuzzy values of outcome decision weights and generalised evaluations of prospect value. The choice of optimal prospect is not difficult because it reduces to a simple comparison of single-sided trapezoidal fuzzy numbers.

The present research uses equal intervals for modelling uncertainties of the values of probability weights. Those intervals are symmetric with regard to the corresponding values of the weighting function. Actually, these intervals can be different for different weights and non-symmetrical with regard to the values of the weighting function.

The validation of the proposed technique is based on the fact that in cumulative prospect theory the sum of prospect decision weights is not required to be equal to 1, which is the case in the rank-dependent utility theory. Due to that, the use of fuzzy weights of probabilities only leads to the fuzziness of resulting evaluations and does not break the conceptual fundamentals of the theory.

References

[1] A. Tversky and D. Kahneman, "Advances in prospect theory: cumulative representation of uncertainty," *Journal of Risk and Uncertainty*, vol. 5, issue 4, pp. 297–327, Oct. 1992. https://doi.org/10.1007/BF00122574

[2] J. von Neuman and O. Morgenstern. *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

[3] W. Edwards, "The Theory of Decision Making," *Psychological Bulletin,* vol. 51, no. 4, pp. 380–417, 1954. https://doi.org/10.1037/h0053870

[4] W. Edwards, "Subjective Probabilities Inferred from Decisions," *Psychological Review*, vol. 69, pp. 109–135, 1962.

[5] M. Allais, "Le comportement de l′homme rationell devant le risqué: Critique des postulats et axioms de l′école américaine". *Econometrica*, vol. 21, pp. 503–546, 1953. https://doi.org/10.2307/1907921

[6] M. J. Machina, "Choice under Uncertainty: Problems Solved and Unsolved," *Perspectives*, vol. 1, no. 1, pp. 121–154, 1987. https://doi.org/10.1257/jep.1.1.121

[7] M. H. Birnbaum, "The Paradoxes of Allais, Stochastic Dominance, and Decision Weights," in J. C. Shanteau, B. A, Mellers & D. A. Schum Eds. *Decision Science and Technology. Reflections on the Contributions of Ward Edwards*, Springer US, 1999. https://doi.org/10.1007/978-1-4615-5089-1_3

[8] G. Wu and R. Gonzalez, "Common Consequence Condition in Decision Making under Risk," *Journal of Risk and Uncertainty*, vol. 16, pp. 115–139, 1998. https://doi.org/10.1023/A:1007714509322

[9] H. Fehr-Duda and T. Epper, "Probability and Risk: Foundation and Economic Implications of Probability Dependent Risk Preferences," *The Annual Review of Economics,* vol. 4, pp. 567–593, 2012. https://doi.org/10.1146/annurev-economics-080511-110950

[10] P. J. H. Schoemaker, "The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations," *Journal of Economic Literature*, vol. 20, no. 2, pp. 529–563, Jun. 1982.

[11] E. U. Weber, "From Subjective Probabilities to Decision Weights: The Effect of Asymmetric Loss Function on the Evaluation of Uncertain Outcomes and Events," *Psychological Bulletin*, vol. 15, no. 2, pp. 228–242, March 1994. https://doi.org/10.1037/0033-2909.115.2.228

[12] J. Quiggin, "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization,* vol. 3, pp. 323–343, Dec. 1982. https://doi.org/10.1016/0167-2681(82)90008-7

[13] D. Kahneman and A. Tversky, "A Prospect Theory," *Econometrica*, vol. 47, no. 2, pp. 263–291, March 1979. https://doi.org/10.2307/1914185

[14] P. Wakker, *Prospect Theory for Risk and Ambiguity,* Cambridge University Press, 2010. https://doi.org/10.1017/CBO9780511779329

[15] D. Prelec, "The Probability Weighting Function", *Econometrica*, vol. 66, no. 3, pp. 497–527, May 1998. https://doi.org/10.2307/2998573

Oļegs Užga-Rebrovs is a Professor in the Faculty of Economics, Rezekne Higher Education Institution (Latvia). He received his Doctoral degree in Information Systems from Riga Technical University in 1994. His research interests include different approaches to processing incomplete, uncertain and fuzzy information, in particular, fuzzy sets theory, rough set theory, as well as fuzzy classification and fuzzy clustering techniques and their applications in bioinformatics.
Address: Rezekne Academy of Technologies, 90 Atbrīvošanas aleja, Rezekne LV-4600, Latvia.
E-mail: ushga@ru.lv.

Gaļina Kuļešova is Research Scientist in the Faculty of Computer Science and Information Technology, Riga Technical University (Latvia). She received her M. Sc. degree in Decision Support Systems from Riga Technical University. Her current research interests include artificial neural networks, data mining, ontology engineering, classification methods and bioinformatics.
Address: Institute of Information Technology, Riga Technical University, Kaļķu iela 1, Riga LV-1658, Latvia.
E-mail: galina.kulesova@ rtu.lv.