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# CHARACTERISTIC PROPERTIES OF NONLINEAR PARAMETRIC OSCILLATIONS OF FLEXIBLE ELEMENTS

# LOKANO ELEMENTU NELINEĀRO PARAMETRISKO SVĀRSTĪBU ĪPATNĪBAS

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## 1. Introduction

Flexible elements (belts, cables, guy ropes, filaments, strings, etc.) are widely used in machines and devices for various practical purposes (belt and chain transmissions, vibrating belts of vibromixers, guy ropes for stabilization of motion of vibromachine tool in prescribed direction, etc) [1, 2]. Lateral parametric vibrations of flexible elements, which can occur during the operation of machine, are extremely detrimental. They give rise to additional dynamic loading, which encourages wearing and failure of flexible elements.
Spectrum of resonance lateral oscillations of flexible elements may be sufficiently dense (simple parametric resonances, combination resonances). Besides, geometrical and physical nonlinearity of flexible element can result in pulling of resonant oscillations and further widening of dangerous frequency ranges. In such conditions system's tuning away from resonance frequencies remains problematic. Therefore in many cases designer had to come to a compromise decision, allowing excitation in machine of one or several (the least intensive) resonance regimes of flexible elements. And therefore it is very important on preliminary stage of designing to evaluate the failure danger of flexible element in each resonant regime which can occur in the system under parametric excitation. This evaluation ought to be based on comparison of different resonant regimes using real criteria (peak values of resonant displacements and tensile forces in flexible element, width of frequency intervals of resonant regimes, etc.).
There is no complete theoretical basis for such criteria in scientific literature. Most of known works on the non-linear oscillations of flexible elements are concerned with the analysis of free vibrations (e.g., [3-5]). But cases of parametric excitation are usually considered in application only to the first lower mode of string's lateral oscillations [6-8]. This paper seeks to compare non-linear properties of different lateral modes of forced and parametric oscillations of flexible element (string). Results of this research will make it possible to form

real criteria for the evaluation of failure danger of flexible element on different parametric resonant regimes.

## 2. Dynamic model

Transverse oscillations of taut flexible element (thread) under parametric excitation are considered (Fig. 1). Parametric excitation is caused by periodic variation in time of axial tension force of the flexible element.

In forming of differential equation of oscillations some assumptions are made. It is supposed, that stiffness in bending of flexible element is negligible in comparison with its stiffness in tension, but weight of flexible element is ignorable in comparison with axial prestressing force $T_0$. Besides, it is considered that oscillations are performed in one plane, which runs along the centre line of a non-deformed flexible element.

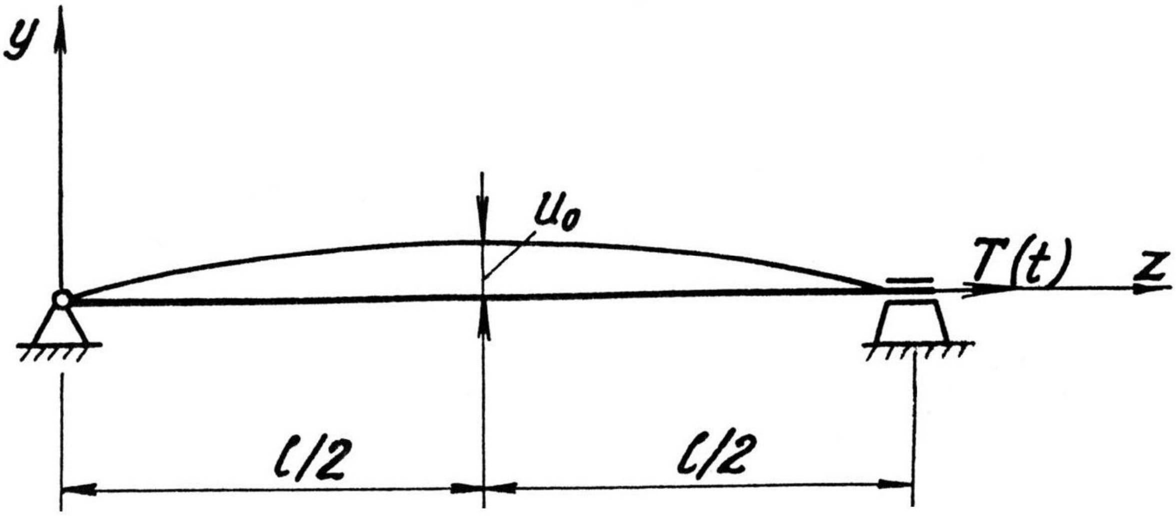


Fig. 1. Model considered in dynamic analysis

Taking the direction of the co-ordinate axis *z* along this centre line, the differential equation for transverse vibrations of flexible element can be stated as follows [2, 5]:

$$T_0(1+\mu\sin\Omega t)[1+f(\varepsilon)](1+b_1\frac{\partial}{\partial t})\frac{\partial^2 y}{\partial z^2}-b_2\frac{\partial y}{\partial t}-$$
$$-\rho[1+\frac{1}{2}(\frac{\partial y}{\partial z})^2]\frac{\partial^2 y}{\partial t^2}=0\,, \tag{1}$$

where $T_0$ is the prestressing force of flexible element; $\mu$ and $\Omega$ are the non-dimensional amplitude and the frequency of parametric excitation; $b_1$ and $b_2$ are the coefficients of internal and external friction; $y$ is the lateral displacement of the flexible element's cross-section with the co-ordinate $z$.

The functional $f(\varepsilon)$ in Eq. (1) takes into account additional tension caused by elastic deformation of flexible element during its oscillations (physical non-linearity). The elongation $\varepsilon$ of flexible element can be determined by formula [5]:

$$\varepsilon = \frac{1}{2l}\int_0^l (\frac{\partial y}{\partial z})^2 dz\,, \tag{2}$$

where $l$ is the length of flexible element.

The relationship between axial stress σ in flexible element and its elongation ε can be approximately described by the expression

$$\sigma = E\varepsilon - \beta\varepsilon^3\,, \tag{3}$$

where $E$ is the elasticity modulus of material; β is the coefficient of non-linearity. In this case the functional $f(\varepsilon)$ can be expressed in the following form

$$f(\varepsilon) = \frac{EA}{2T_0 l}\int_0^l (\frac{\partial y}{\partial z})^2 dz - \frac{\beta A}{8T_0 l^3}[\int_0^l (\frac{\partial y}{\partial z})^2 dz]^3\,, \tag{4}$$

where $A$ is the cross-section area of flexible element.

Therefore an increment in tension is caused by integral elongation of flexible element and is independent of co-ordinate $z$. Non-linear term $[1+\frac{1}{2}(\frac{\partial y}{\partial z})^2]$ of equation (1) takes into account geometrical non-linearity of flexible element [5].

In the case studied here the end boundary conditions are as follows:

$$y(z=0,t)=0; \qquad y(z=l,t)=0\,. \tag{5}$$

Equation (1), subject to the expressions (2) – (5), was solved on an analogue-digital computer system predominantly set up for the solution of complex non-linear dynamics problems [2]. The integration of non-linear differential equations is carried out on the high-speed analogue part of the computer system, but control over the programming of the analogue part and data processing is executed by the digital part. The methods of mathematical simulation and the operational principle of the computer system are described in more detail in references [2, 9]. The quantitative estimation of accuracy in analogue-digital simulation was carried out by the solution of test examples and particular engineering problems [2, 10, 11].

**3. Analysis of parametric oscillations of flexible element**

As is known [12, 13], parametric resonance of flexible element occur under periodic pulsation of tensile force $T$ with frequency Ω which fall in the vicinity of critical frequencies

$$\Omega = \frac{2\omega_S}{e} \quad \text{or} \quad \Omega = \frac{|\omega_s \pm \omega_k|}{e}\,, \tag{6}$$

where $\omega_S$ and $\omega_k$ are the natural frequencies with ordinal numbers $S$ and $k$ of lateral oscillations of flexible element; $e$ = 1, 2, 3 … is the order of parametric resonance. The first formula describes the condition of a simple parametric resonance, but the second formula – condition of a combination parametric resonance.

As an example, Fig. 2 shows the diagram of parametric resonance zones on the plane of parameters $\mu$ and $\eta = \Omega/\omega_1$ (zones are section-lined and denoted with symbols $2\omega_S$ or $\omega_S + \omega_k$). Only main zones ($e$ = 1) corresponding to the three first natural frequencies $\omega_1$, $\omega_2$ and $\omega_3$ are shown. The diagram is constructed assuming $b_1\omega_1 = 0.003$ (it is the rough level of losses in real flexible elements). Combination resonances of difference type $\frac{\omega_S - \omega_k}{S}$ for the system under study were not observed.

Objective conclusion on failure danger of flexible element in one or another parametric regime can be made by comparison of corresponding tensile forces.

### 3.1. Simple parametric oscillations

Simple parametric oscillations have been analyzed at first. In this case main system's parameters reduced to the dimensionless form have been varied within the limits: $T_0/EA$ = $= (0.5 \div 5)\cdot 10^{-4}$; $b_1\omega_1 = 0.002 \div 0.018$; $\mu = 0 \div 0.5$. Experimental points have been located within this space of parameters in accordance with the uniform distribution design [14, 15] (for each variable parameter ten value levels have been selected). But value of the factor $\eta = \Omega/\omega_1$ has been chosen during the simulation provided system's tuning to the resonance frequency of considered order. Values of other system's parameters remained constant ($\rho lg/EA = 5.7\cdot 10^{-6}$; $\beta/E = 0.36$). A body of research is limited with resonance regimes corresponding to the three lowest natural frequencies of flexible element ($\omega_1$, $\omega_2$, $\omega_3$).

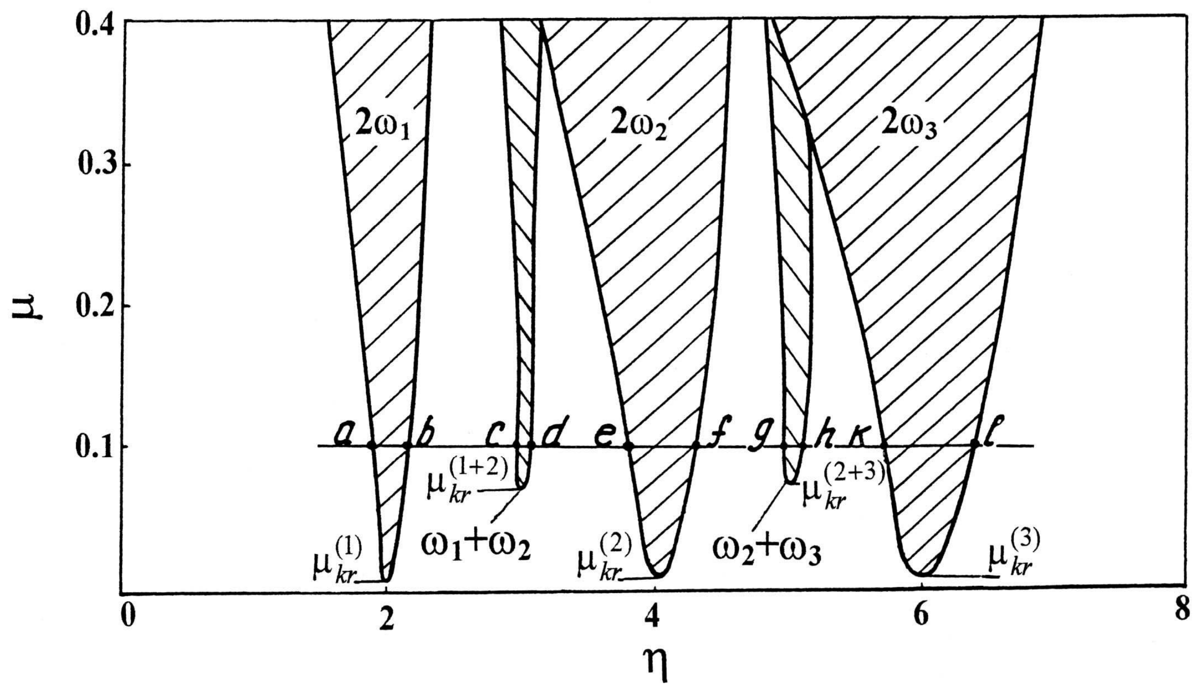


Fig. 2. Zones of excitation of parametric lateral oscillations of flexible element

During the simulation peak values of dimensionless displacements $u_0^{(S)}/l$ of flexible elements (in antinodal point of resonant mode) and corresponding dynamic components of tensile force $T_d^{(S)}/T_0$ have been determined for each combination of factors $T_0/EA$, $b_1\omega_1$ and $\mu$. The results are presented in Table 1.

It is seen from the analysis of table data that the relationship between peak values of lateral displacements of flexible element on simple parametric resonances (regardless of numerical values of system's parameters) on the average is as follows:

$$u_0^{(1)} : u_0^{(2)} : u_0^{(3)} = 0.023 : 0.0125 : 0.008 \approx 1 : \frac{1}{1.85} : \frac{1}{2.9} .$$

But dynamic components of tensile force (in the three first resonant regimes) change their values in accordance with the proportion

$$T_d^{(1)} : T_d^{(2)} : T_d^{(3)} = 8.2 : 8.8 : 8.65 \approx 1 : 1.07 : 1.05 .$$

Thus, with the rise of the ordinal number $S$ of the mode of parametric regime the tensile forces in flexible element slightly increase (in spite of sufficient reduction of lateral displacements $u_0^{(S)}$).

Table 1. Lateral displacements $u_0^{(S)}/l$ and dynamic components of tensile force $T_d^{(S)}/T_0$ in simple parametric resonant regimes of flexible element

| Ordinal number | $\frac{T_0}{EA}\cdot 10^4$ | $b_1\omega_1 \cdot 10^3$ | μ | $u_0^{(1)}/l$ | $T_d^{(1)}/T_0$ | $u_0^{(2)}/l$ | $T_d^{(2)}/T_0$ | $u_0^{(3)}/l$ | $T_d^{(3)}/T_0$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 8 | 0.45 | 0.024 | 2.30 | 0.013 | 2.50 | 0.008 | 2.40 |
| 2 | 2 | 3 | 0.10 | 0.036 | 12.90 | 0.019 | 14.0 | 0.0125 | 13.70 |
| 3 | 1.5 | 10 | 0.50 | 0.029 | 11.20 | 0.015 | 12.0 | 0.010 | 11.80 |
| 4 | 3 | 14 | 0.05 | 0.006 | 0.30 | 0.003 | 0.32 | 0.002 | 0.31 |
| 5 | 4.5 | 16 | 0.25 | 0.011 | 0.55 | 0.006 | 0.57 | 0.004 | 0.57 |
| 6 | 0.5 | 12 | 0.20 | 0.014 | 7.80 | 0.0075 | 8.50 | 0.005 | 8.30 |
| 7 | 1 | 4 | 0.40 | 0.042 | 35.20 | 0.023 | 38.0 | 0.015 | 37.20 |
| 8 | 2.5 | 18 | 0.35 | 0.009 | 0.65 | 0.005 | 0.70 | 0.003 | 0.68 |
| 9 | 3.5 | 2 | 0.30 | 0.040 | 9.10 | 0.021 | 9.90 | 0.014 | 9.60 |
| 10 | 4 | 6 | 0.15 | 0.020 | 2.00 | 0.011 | 2.10 | 0.007 | 2.10 |
| Average values | | | | **0.023** | **8.20** | **0.0125** | **8.80** | **0.008** | **8.65** |

As an illustration of the above-mentioned relationships Fig. 3 shows the amplitude-frequency characteristic (AFC) of parametric oscillations of flexible element for the case $\mu = 0.1$, $T_0/EA = 2\cdot10^{-4}$ and $b_1\omega_1 = 0.003$. Dimensionless displacements $u_0/l$ (in antinodal points of each resonant mode) and dynamic components of tensile force $T_d/T_0$ are projected as amplitudes on these AFC. As it is seen from the AFC presented, the following relationships between peak values of lateral displacements and tensile forces in flexible element hold true in simple parametric resonant regimes:

$$u_{01}^{(A)} : u_{02}^{(B)} : u_{03}^{(C)} = 0.036 : 0.019 : 0.0125 \approx 1 : \frac{1}{1.85} : \frac{1}{2.9}$$

and $$T_{d1}^{(A)} : T_{d2}^{(B)} : T_{d3}^{(C)} = 12.9 : 14 : 13.7 \approx 1 : 1.08 : 1.06 .$$

Thus, under the parametric excitation the rise of ordinal number *S* of the resonant mode causes a certain increase of tensile forces in flexible element. Therefore a danger of failure of flexible element on the second and third parametric resonant regimes is slightly higher than on the first one.

### 3.2. Combination parametric resonances

Combination parametric resonance earlier has been analysed in application to rod structures [16, 17]. It was shown that peak values of displacements on combination and simple parametric resonances are comparable. On this base the conclusion on approximately equal danger extent of combination and simple parametric oscillation in rod structures has been made (in spite of relative narrowness of zones of combination parametric resonances).

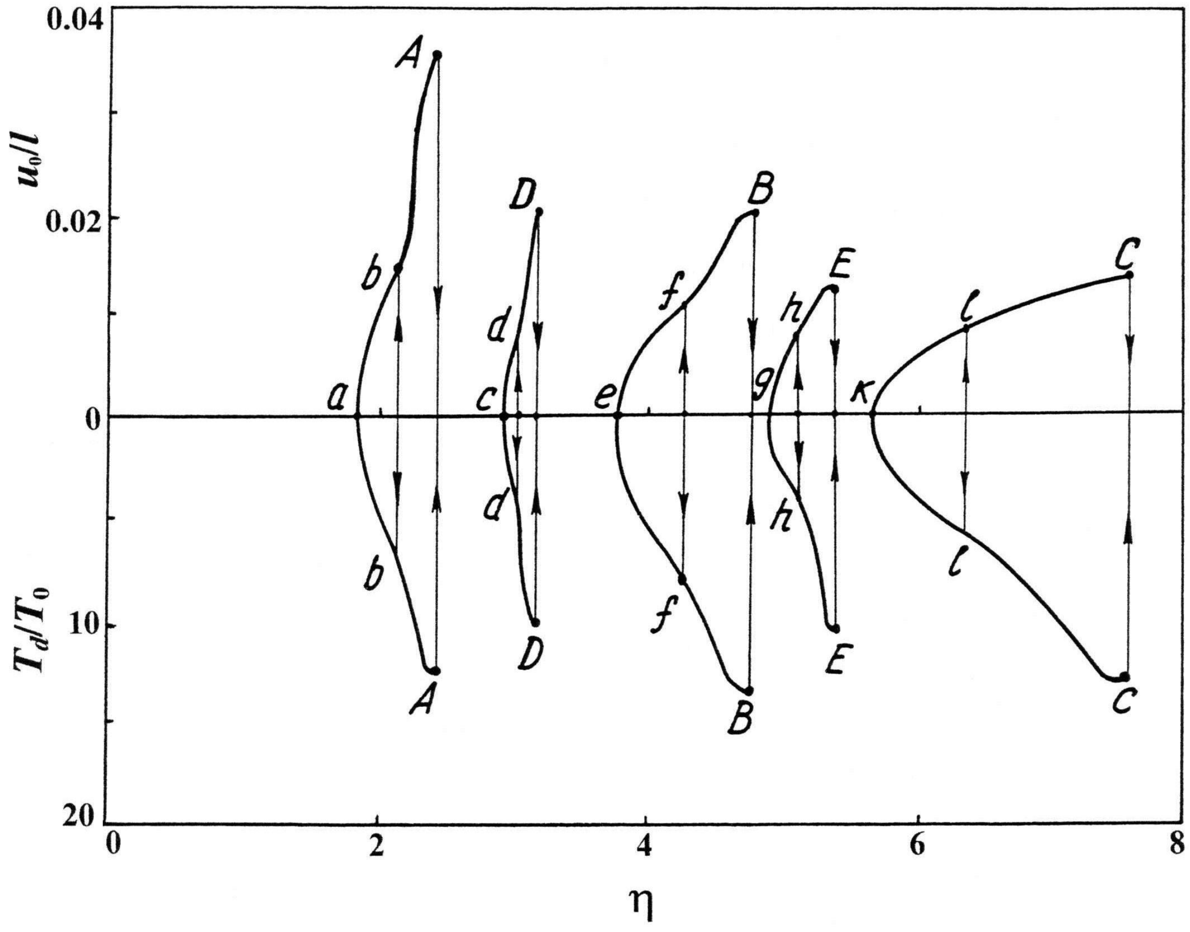


Fig. 3. AFC of parametric oscillations of flexible element (for lateral displacements $u_0/l$ and tensile forces $T_d/T_0$)

Therefore it has been found expedient to evaluate a danger extent of combination parametric oscillations of flexible elements. At first conditions of excitation of combination parametric resonances are considered. As it is follows from the diagram of parametric instability zones (see Fig. 2), zones of combination resonance are sufficiently more narrow (by excitation frequency Ω) than zones of simple parametric resonance. For example, under $\mu = 0.2$ the width of zones $2\omega_1$ and $2\omega_2$ is equal to $\Delta\eta^{(1)} = 0.42$ and $\Delta\eta^{(2)} = 0.83$ correspondingly. But at the same time the width of combination parametric zone ($\omega_1 + \omega_2$) is about only $\Delta\eta^{(1+2)} = 0.1$. Besides, excitation of combination parametric oscillations is possible only under more high pulsation parameter $\mu$ of tensile force $T$ than it is necessary for excitation of simple parametric resonances. For example, under the given value of $b_1\omega_1$ the threshold of parametric excitation for the zone $2\omega_1$ is equal to $\mu_{kr}^{(1)} = 0.0064$ , but for the zone ($\omega_1+\omega_2$) - $\mu_{kr}^{(1+2)} = 0.07$ (approximately is ten times higher).

Nevertheless, conclusion on real danger of combination parametric oscillations of flexible element can be made only on the base of evaluation of its lateral displacements and corresponding values of tensile forces. These data are given in Table 2.

Table 2. Lateral displacements $u_0^{(S+k)}/l$ and dynamic components of tensile force $T_d^{(S+k)}/T_0$ in combination parametric resonant regimes of flexible element

| Ordinal number | $\frac{T_0}{EA}\cdot 10^4$ | $b_1\omega_1 \cdot 10^3$ | $\mu$ | $u_0^{(1+2)}/l$ | $T_d^{(1+2)}/T_0$ | $u_0^{(2+3)}/l$ | $T_d^{(2+3)}/T_0$ |
|---|---|---|---|---|---|---|---|
| 1 | 5 | 8 | 0.45 | 0.014 | 1.90 | 0.008 | 1.90 |
| 2 | 2 | 3 | 0.10 | 0.020 | 10.50 | 0.011 | 11.0 |
| 3 | 1.5 | 10 | 0.50 | 0.016 | 9.80 | 0.009 | 9.10 |
| 4 | 3 | 14 | 0.05 | 0.004 | 0.25 | 0.002 | 0.24 |
| 5 | 4.5 | 16 | 0.25 | 0.006 | 0.41 | 0.003 | 0.43 |
| 6 | 0.5 | 12 | 0.20 | 0.008 | 6.30 | 0.0045 | 6.20 |
| 7 | 1 | 4 | 0.40 | 0.024 | 28.50 | 0.013 | 28.40 |
| 8 | 2.5 | 18 | 0.35 | 0.005 | 0.50 | 0.003 | 0.50 |
| 9 | 3.5 | 2 | 0.30 | 0.021 | 7.40 | 0.012 | 7.50 |
| 10 | 4 | 6 | 0.15 | 0.012 | 1.50 | 0.007 | 1.50 |
| Average values | | | | **0.013** | **6.60** | **0.007** | **6.65** |

As it is seen from the analysis of table data, the relationships between lateral displacements and tensile forces in flexible element on combination parametric resonances are as follows:

$$u_0^{(1+2)} : u_0^{(2+3)} = 0.013 : 0.007 \approx 1.86 : 1$$

and

$$T_d^{(1+2)} : T_d^{(2+3)} = 6.6 : 6.65 \approx 1 : 1.007 .$$

Moreover, numerical values of tensile forces $T_d^{(s+k)}$ are about on 25 – 30% smaller than that ones realized on simple parametric resonances. These characteristic properties of oscillations are presented graphically on the AFC (see Fig. 3).

Thus, the results of mathematical simulation show that combination parametric resonance of flexible element is less dangerous than simple parametric ones. This is caused by the more narrow frequency interval and higher threshold of parametric excitation as well by the lower level of tensile stresses on combination resonances.

The practicability of the results obtained by the analogue-digital simulation was verified by experiments. As an object of study a uniform rubber cord with the length $l$ = 1.5 m and linear density $\rho$ = 0.0415 kg/m (in unloaded condition) was used. Parametric oscillations of the cord are excited with the aid of electrodynamic vibration-testing machine (model ВЭДС-10А). Parameters of oscillations of flexible element are measured by vibration meter (model SDM–162) using standard and specialized piezoelectric transducers.

During the experiments the rates of changing of peak values of lateral displacements of flexible element and tensile forces under the sequential excitation of parametric resonant oscillations corresponding to natural frequencies with more and more high order ($\omega_1$, $\omega_2$, $\omega_3$) were studied. Experimental relationships between lateral displacements and tensile forces have shown close agreement with the results of mathematical simulation.

## 4. Conclusions

Characteristic properties of resonant lateral oscillations of flexible element under parametric excitation are studied. As the result of this research an evaluation of failure danger of flexible element in different resonant regimes is made. The main conclusions are as follows.

(1) Under the parametric excitation the rise of ordinal number $S$ of simple parametric regime causes a certain increase of tensile forces and dynamic stress in flexible element. Therefore a danger of failure of flexible element on the second and third parametric resonant regimes is slightly higher than on the first one.

(2) Combination parametric oscillations of flexible element are less dangerous than simple parametric ones. This is caused by the more narrow frequency interval and higher threshold of parametric excitation as well by the lower level of tensile stresses on combination resonances.

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***Beresņevičs V. Lokano elementu nelineāro parametrisko svārstību īpatnības***
*Aplūkoti mašīnu un mehānismu lokano elementu (siksnu, kabeļu, tauvu, stīgu u.c.) šķērssvārstības pie parametriskās ierosmes. Matemātiski uzdevums ir formulēts kā lokana elementa parametrisko svārstību parciālais diferenciālvienādojums, turklāt tiek ņemta vērā lokana elementa ģeometriskā, statiskā un fiziskā nelinearitāte. Apskatāmajā sistēmā tiek novērtēta dažādu iespējamo rezonanses režīmu (parastas parametriskās rezonanses, kombinacionālās rezonanses) bīstamības pakāpe, izmantojot matemātiskās modelēšanas metodes. Šīs novērtēšanas pamatā ir režīmu salīdzināšana pēc ierosmes frekvenču intervālu platuma, transversālo pārvietojumu un stiepējspēku maksimālajiem lielumiem u.c. Parādīts, ka lokano elementu kombinacionālās parametriskās svārstības ir mazāk bīstamas par svārstībām parastos parametriskos rezonanses režīmos. Teorētiskās analīzes rezultāti apstiprināti eksperimentāli.*

***Beresnevich V. Characteristic Properties of Nonlinear Parametric Oscillations of Flexible Elements***
*Lateral oscillations of flexible elements (belts, cables, guy ropes, strings, etc.) in machines and devices under parametric excitation are studied. Mathematically the problem is presented as a partial differential equation describing parametric oscillations of flexible element with due account of its geometrical, static and physical nonlinearities. By the mathematical simulation the evaluation of danger extent of different resonant regimes which can occur in the system (simple parametric resonances, combination resonances) is made. This evaluation is based on comparison of excitation frequency intervals of resonant regimes, peak values of resonant displacements and tensile forces in going from lower to higher vibration mode, etc. It is shown that combination parametric oscillations of flexible element are less dangerous than simple parametric ones. The results of theoretical study are confirmed by experiments with a physical model.*

***Бересневич В. Особенности нелинейных параметрических колебаний гибких элементов***
*Рассмотрены поперечные колебания гибких элементов машин и механизмов (ремней, тросов, растяжек, струн и др.) при параметрическом возбуждении. Математически задача сводится к дифференциальному уравнению в частных производных, описывающему параметрические колебания гибкого элемента с учетом его геометрической, статической и физической нелинейностей. На основе математического моделирования колебаний дана оценка степени опасности различных резонансных режимов (простые параметрические резонансы, комбинационные резонансы), потенциально возможных в рассматриваемой системе. В основе такой оценки – сопоставление различных режимов по ширине частотных интервалов их возбуждения, пиковым значениям поперечных перемещений и растягивающих усилий в гибком элементе. Показано, что при прочих равных условиях комбинационные параметрические колебания гибкого элемента менее опасны простых параметрических колебаний. Результаты теоретического анализа подтверждены экспериментально.*