Advanced Portfolio Optimization Using Copula and GARCH Models: A High-Performance Computing Approach for the European Stock Market

Jegors Fjodorovs
Institute of Applied Mathematics
Riga Technical University
Riga, Latvia
Jegors.Fjodorovs@rtu.lv

Andrejs Matvejevs
Institute of Applied Mathematics
Riga Technical University
Riga, Latvia
Andrejs.Matvejevs@rtu.lv

Anastasia Vasiljeva
Institute of Applied Mathematics
Riga Technical University
Riga, Latvia
Anastasija.Vasiljeva_7@edu.rtu.lv

***Abstract*—This paper explores advanced portfolio optimization strategies using copula and GARCH models to enhance risk management and profitability in the European stock market. By utilizing high-performance computing (HPC) to conduct extensive simulations on approximately 10,000 portfolios, the study compares the effectiveness of various copula-GARCH models against traditional approaches, such as the mean-variance model. The most effective configuration—identified as a Student's copula with marginal Student's distribution and an eGARCH model—was employed to simulate returns and construct optimal portfolios that minimize Conditional Value at Risk (CVaR). The scalability and robustness of this approach offer valuable insights into its practical applications for portfolio management.**

Keywords—copula, GARCH, CVaR, portfolio optimization, European stock market, high-performance computing, simulations

I. Introduction

Portfolio optimization is a fundamental aspect of financial management, focused on maximizing returns while minimizing risk. Traditional approaches, such as the mean-variance optimization introduced by Markowitz (see, for example [1] and [2]), have been widely adopted but often struggle to accurately capture the complexities of financial markets, especially when dealing with non-linear dependencies and heavy-tailed distributions [3].

In recent years, the use of copula and GARCH (Generalized Autoregressive Conditional Heteroskedasticity) models has gained prominence due to their ability to better model dependencies between asset returns and account for varying volatility [4]. These models are particularly valuable in risk management, where a deep understanding of the joint distribution of asset returns is essential.

In a financial portfolio, assets are rarely independent of one another; the return on one asset is often correlated with the return on another. For example, stocks within the same industry or region tend to move together. Accurately modelling these dependencies and correlations requires a deep understanding of the joint distribution of asset returns. This allows risk managers to comprehensively assess how different assets interact, particularly under extreme market conditions, leading to improved risk assessment, portfolio optimization, and stress testing. This paper presents the simulation algorithm for EURO STOXX 50 Index stocks using copula-GARCH models, compares it to traditional optimization techniques, and evaluates their performance outcomes.

II. Traditional Methods

A. Markowitz Approach

The Markowitz approach [1], also known as mean-variance analysis, is a fundamental component of Modern Portfolio Theory. It provides a framework for achieving a higher expected portfolio return for a given level of risk, or conversely, minimizing risk for a given expected return. The expected portfolio return is calculated as:

$$E(R_p) = \sum_{i=1}^{n} w_i * E(R_i)\,, \tag{1}$$

where:

- w_i is the weight of the *i-th* asset in the portfolio,
- $E(R_i)$ is the expected return of the *i-th* asset.

This formula allows for the efficient allocation of assets to optimize returns while managing risk.

In the Markowitz model, portfolio risk σ_p is measured by the standard deviation of returns, where the portfolio variance is calculated as:

$$Var(R_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i * w_j * Cov(R_i, R_j) \tag{2}$$

The portfolio risk σ_p is then:

$$\sigma_p = \sqrt{Var(R_p)}$$

While Markowitz's portfolio optimization is a cornerstone of financial theory, its practical application is limited by several assumptions and challenges. These include reliance on historical data, sensitivity to input estimates, the assumption of normality, and the neglect of extreme events and multi-period investment horizons. Such limitations underscore the need for more advanced models that better capture the complexities of real-world financial markets [5, 6].

B. GARCH

The GARCH model [4] is a generalization of the ARCH model. A characteristic of the ARCH process is that its

979-8-3315-3383-0/24/$31.00 ©2024 IEEE

conditional standard deviation σ_t or volatility, is a continuously varying function of the previous values of the square of the process. On the other hand, GARCH is a generalization in the sense that the variance σ_t^2, or the squared volatility, is allowed to depend on the previous squared fluctuations as well as on the previous squared values of the process itself [1].

The GARCH(p,q) process can be represented by the formula:

The GARCH model [4] is a generalization of the ARCH model, which models time-varying volatility in financial time series. In an ARCH process, the conditional standard deviation σ_t, or volatility, is a function of previous squared values of the process. GARCH extends this by allowing the conditional variance σ_t^2 to depend not only on past squared fluctuations but also on past values of the variance itself [7].

The GARCH(p,q) process is represented by the following formulas:

$$X_t = \sigma_t Z_t \,, \tag{3}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i X_{t-i}^2 + \sum_{j=1}^{p} \beta_i \sigma_{t-j}^2 \,, \tag{4}$$

where:

- t is time,
- p, q are the orders of the ARCH and GARCH components, respectively,
- α_0 is a constant,
- σ_t^2 represents the conditional variances,
- α_i are the ARCH model parameters,
- β_i are the GARCH model parameters,
- X_t is the ARCH process,
- Z_t is a white noise process.

The GARCH model provides crucial insights into the volatility and risk of financial instruments, enabling more accurate risk assessment for portfolios. Additionally, it addresses the limitation of the Markowitz model, which assumes constant volatility, offering a more realistic view of market dynamics.

C. Copulas

In modern statistics [8] and data analysis, copulas, introduced by Sklar are gaining popularity due to their flexibility and accuracy in modeling the distribution of multivariate random variables. Their application extends across various fields, including finance, insurance, econometrics, artificial intelligence, and even climatology. This growing interest has also attracted the attention of probability theorists.

Copulas are particularly useful for modeling dependencies between random variables when traditional methods, such as Pearson correlation, fall short. Pearson correlation only captures linear relationships, whereas copulas can account for more complex, non-linear dependencies and the rank of the random variables [7].

An n-dimensional copula is a function C:$[0,1]^n \rightarrow [0,1]$ that satisfies the following properties [9]:

- $\forall u \in [0,1]$, $C(1, \ldots, 1, u, 1, \ldots, 1) = u$,
- $\forall u \in [0,1]$, $C(u_1, \ldots, u_n) = 0$, if at least one of u_i s equal to zero.
- C is grounded and n-increasing, meaning the C-volume of every box with vertices in $[0, 1]^n$ is positive.

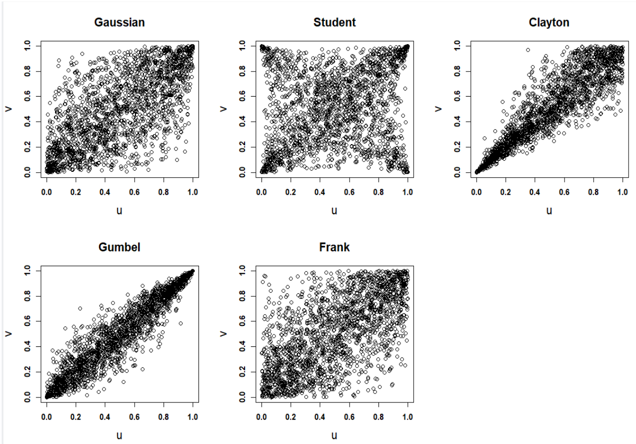

Fig.1. Most popular copula densities in finance

The Copula-GARCH approach enables more realistic simulations of future portfolio performance across various market scenarios. It not only captures the true dependency structure between assets but also reflects the dynamic nature of market volatility. Consequently, this approach enhances stress testing and scenario analysis, thereby improving the robustness of asset allocation decisions [10]. By accurately modeling both the dependency structure and the time-varying nature of risk, Copula-GARCH optimization identifies better diversification opportunities and constructs portfolios that are less susceptible to simultaneous large losses, ultimately improving risk-adjusted returns. Traditional optimization methods may overlook these complexities, leading to suboptimal diversification and increased overall portfolio risk. Therefore, Copula-GARCH investment portfolio optimization is potentially superior for asset allocation, as it more effectively captures the complex, dynamic relationships between assets compared to traditional models. By addressing non-linear dependencies, tail risks, and time-varying volatility, it provides a more accurate and robust framework for portfolio optimization, leading to enhanced risk management and performance under diverse market conditions. In the following sections, we will discuss the construction of the Copula-GARCH algorithm and present the simulation results.

III. Algorithm for Constructing a Portfolio Based on Copula and GARCH

The construction and optimization of a portfolio based on copula and GARCH [4] models can be efficiently accomplished using the following algorithm:

- **Stock selection and obtaining logarithmic returns**: Choose the stocks for the portfolio and compute their logarithmic returns.
- **Testing for stationarity and the arch effect:** Verify the stationarity of the return series and test for the presence of the ARCH effect.
- **GARCH model selection:** Identify the appropriate GARCH model to capture the time-varying volatility in the return series.
- **Selection of the marginal distribution:** Choose the marginal distribution that best fits the individual asset returns.
- **Copula selection from transformed standardized GARCH residuals:** Select a copula function based on the transformed standardized GARCH residuals to model the dependencies between assets.
- **Simulating returns from the copula function:** Generate simulated returns using the chosen copula function.
- **Portfolio optimization using CVAR minimization:** Optimize the portfolio based on the simulated returns by minimizing Conditional Value at Risk (CVaR).

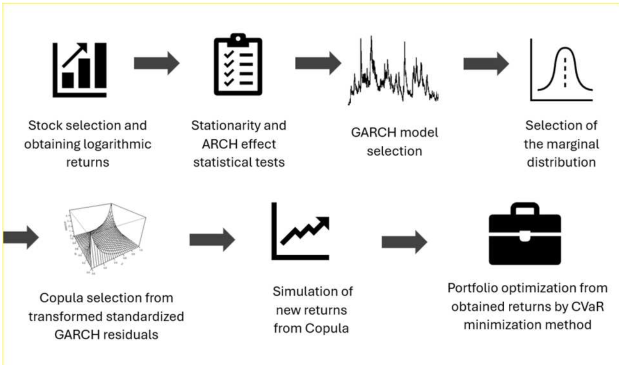

Fig.2. Algorithm for copula-GARCH model-based portfolio optimization

The following algorithm was implemented using R Studio and Riga Technical University's high-performance computing (RTU HPC) infrastructure to extend the portfolio optimization method to 10,000 portfolios. The RTU HPC cluster comprises 34 computing nodes for job execution and one head node responsible for cluster management. All nodes are interconnected via a high-speed InfiniBand network. Each compute node is equipped with two x86_64 architecture processors (CPUs), and some nodes also feature 2 or 4 Nvidia Tesla graphical accelerators (GPUs). The cluster architecture is heterogeneous, combining nodes of varying generations and technical specifications [11]. Consequently, 100,000 simulations from the copula function were performed for each stock to achieve maximum likelihood results.

For each portfolio, 5 stocks were randomly selected from the EURO STOXX 50 index. Each portfolio was optimized using three models: the Markowitz model, the copula-GARCH model with a CVaR minimization approach, and a minimal CVaR model based on historical data. The historical data used for optimization and the estimation of copula and GARCH parameters covered the period from January 1, 2014, to December 31, 2021.

IV. Copula and GARCH Model Selection for EURO STOXX 50 Index Stocks

To identify the most suitable copula for modeling the dependence structure between asset returns in the EURO STOXX 50 index [12], a comprehensive analysis of all possible pairs of the 50 constituent stocks was conducted. The process involved the following steps:

1. ***Data Collection***

   Historical daily closing prices for all 50 stocks in the EURO STOXX 50 index were collected and transformed into log returns to ensure stationarity.

2. ***Pairwise Analysis***

   Each of the 1,225 possible pairs of stocks was analyzed to determine the best-fitting copula. The copula families considered included Gaussian, Clayton, Gumbel, Frank, and Student's t-copula.

3. ***Goodness-of-Fit Testing***

   For each pair, goodness-of-fit tests, such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC), were used to evaluate the copula models. These tests penalize for model complexity and assist in selecting a model that balances fit and simplicity.

4. ***Selection Criteria***

   The copula that most frequently emerged as the best fit across all pairs was chosen as the most suitable for the entire dataset. This process revealed that the Student's t-copula was the most appropriate, indicating its effectiveness in capturing the tail dependence and heavy tails characteristic of the joint distribution of stock returns.

TABLE I. BISELECTCOP() FUNCTION RESULTS FOR THE STOCKS

Clayton	Frank	Gaussian	Gumbel	Student
-	3	2	44	1032

To model the conditional volatility of returns, various GARCH models were considered. The process for selecting the most suitable GARCH model involved the following steps:

1. ***Data Collection***
   Historical daily closing prices of the EURO STOXX 50 index were used to compute the index returns.

2. ***Model Specification***
   Various GARCH model variants were considered, including the standard GARCH, GJR-GARCH, Integrated GARCH (iGARCH), and Exponential GARCH (eGARCH). The eGARCH model was particularly noted for its ability to capture asymmetries in volatility

3. *Model Evaluation*

The suitability of each GARCH model was assessed using information criteria such as the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) [13]. These criteria help balance model fit and complexity, ensuring that the selected model is both accurate and parsimonious.

The eGARCH model emerged as the most suitable for the EURO STOXX 50 index returns due to its superior performance in capturing asymmetric volatility effects. This model effectively accounts for the phenomenon where negative shocks tend to impact volatility more than positive shocks, a common characteristic in financial markets.

TABLE II. GARCH MODELS RESULTS

GARCH models	AIC	BIC	LogLik
sGARCH	-6.307543	-6.291929	5527.254
eGARCH	-6.356534	-6.337798	5571.146
gjrGARCH	-6.347582	-6.328845	5563.308
iGARCH	-6.308296	-6.295805	5526.913

V. RESULTS AND DISCUSSIONS

The models were tested using data from the year 2022, applying the portfolio weights obtained from the optimization to the selected shares. Profit and loss (PnL) for all portfolios were calculated as of December 29, 2022. The histograms displaying the PnL results are presented in Figure 3.

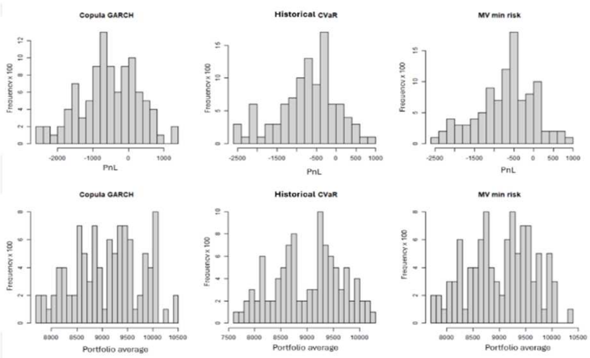

Fig.3. Profit and Loss (PnL) histograms as of December 29, 2022.

The Copula-GARCH models produced more positive values compared to the other two models. Table III provides a summary of the number of instances where each model achieved the highest PnL, along with the minimum, maximum, and average values. Additionally, the table presents the lowest CVaR index and standard deviation for each model.

TABLE III. THE NUMBER OF POSITIVE OUTCOMES FOR EACH MODEL

Model	Max PnL	Max Min	Max Max	Max Mean	Min Std	Min CVaR
t-Copula eGARCH	5712	5071	6371	5805	2507	4301
Historical CVaR	2431	2204	2505	2104	3805	5699
MV min risk	1857	2314	1124	2091	3688	0

In 2022, the market experienced a decline due to political events, providing an opportunity to assess the model's effectiveness in a negative scenario. However, the t-Copula eGARCH investment portfolio construction yielded significantly better results. For instance, Figure 4 illustrates the performance of hypothetical portfolios throughout 2022, where the t-Copula eGARCH portfolio demonstrates greater resilience to market downturns.

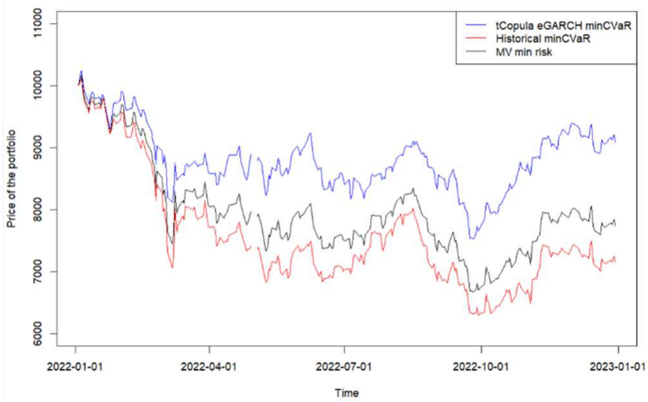

Fig.4. Trajectories of three hypothetical portfolios in 2022.

Another backtesting was conducted using data from the year 2023, a period characterized by positive market dynamics. This allowed for further evaluation of the model's performance under favorable conditions.

TABLE IV. THE NUMBER OF POSITIVE OUTCOMES FOR EACH MODEL

Model	Max PnL	Max Min	Max Max	Max Mean	Min Std	Min CVaR
t-Copula eGARCH	6605	3104	4203	5201	4709	3435
Historical CVaR	2605	2001	4705	4107	2104	6565
MV min risk	790	1401	1092	692	3187	0

As shown in Table IV, the PnL of the copula-GARCH model was higher in 66% of cases compared to the historical CVaR and mean-variance models. However, other performance indicators were not entirely favorable. The minimum CVaR model, constructed using historical data, more frequently achieved the lowest CVaR among the three models and also more often reached the maximum portfolio value.

VI. CONCLUSIONS

This study investigated the effectiveness of copula and GARCH models in portfolio optimization, with a focus on risk management and the composition of portfolios consisting of individual stocks from the European market. Accurate examination of selected parameters and initial conditions is crucial to ensure the model's precision and reliability.

To achieve the desired results, a comprehensive literature review and empirical study were conducted using the R language within the RStudio environment. High-performance computing resources were employed to test the models across a large number of portfolios and to compare the outcomes with traditional models, such as mean-variance and historical approaches, as well as the proposed copula-GARCH model, which combines Student's t-copula with exponential GARCH.

REFERENCES

[1] Markowitz, H.M. Portfolio Selection. The Journal of Finance. 7 (1): 77–91, 1952

[2] Sharpe, W. F. The Sharpe Ratio. The Journal of Portfolio Management, 1994. 21 (1): 49–58. Available from: doi:10.3905/jpm.1994.409501.

[3] Natale, F., P. Optimization With Tail-Dependence and Tail Risk: A Copula Based Approach For Strategic Asset Allocation. Milan, 2006, Available from: doi: http://dx.doi.org/10.2139/ssrn.942275.

[4] Bollerslev, T., Generalized autoregressive conditional heteroscedasticity, Journal of Econometrics Vol. 31, No. 3, pp. 307–327, 1986.

[5] Chopra, V. K., & Ziemba, W. T. (1993). "The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice." Journal of Portfolio Management

[6] Taleb, N. N. (2007). "The Black Swan: The Impact of the Highly Improbable." Random House

[7] McNeil, A., Frey, R., Embrechts, P. *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton and Oxford: Princeton University Press, 2005, 554 pp. ISBN 0-691-12255-5.

[8] Darsow, W., Nguyen, B., Olsen, E. Copulas and Markov processes. Illinois Journal of Mathematics. 1992, 36, 600–642. Available from: doi: 10.1215/IJM/1255987328

[9] Malevergne, Y., Sornette, D. *Extreme Financial Risks: From Dependence to Risk Management*, Berlin: Springer, 2006. 307p. ISBN 978-3-540-27264-9

[10] Cherubini, U., E. Luciano and W. Vecchiato, 2004, Copula Methods in Finance,. John Wiley & Sons, England.

[11] RTU HPC User Manual. https://hpc-guide.rtu.lv/index.html

[12] STOXX. EURO STOXX 50 [online]. stoxx.com, 2024 [viewed 15 May 2024]. Available from: https://stoxx.com/index/SX5E/#

[13] Hamilton, J.D., Time series analysis. Princeton University Press, 1994. 799 p. ISBN 978-0691042893.

[14] Abad, P., Benitob, S., López, C. A comprehensive review of Value at Risk methodologies. The Spanish Review of Financial Economics. 2014, 12(1), pp. 15- 32. ISSN 2173-1268. Available from: doi: http://dx.doi.org/10.1016/j.srfe.2013.06.001.

[15] Alotaibi, T. S., Luciana, D. V., Craven, M. J. The worst case GARCH-copula CVaR approach for portfolio optimisation: Evidence from financial markets, Journal of Risk and Financial Management, 2022. ISSN 1911-8074, MDPI, Basel, Vol. 15, Iss. 10, pp. 1-14, Available from: doi:https://doi.org/10.3390/jrfm15100482.

[16] Bohdalová, M., Greguš, M. PORTFOLIO OPTIMIZATION AND SHARPE RATIO BASED ON COPULA APPROACH. Research Journal of Economics, Business and ict. 2012.

[17] Bouye, E., Salmon, M. Dynamic Copula Quantile Regressions and Tail Area Dynamic Dependence in Forex Markets. [online] Financial Econometrics Research Centre, 2002 [viewed 10 May 2024]. Available from: https://warwick.ac.uk/fac/soc/wbs/subjects/finance/research/wpaperseries/2003/03-121.pdf

[18] Cech, C. Copula-Based Top-Down Approaches in Financial Risk Aggregation. Vienna: University of Applied Sciences, 2006 Available from: doi: http://dx.doi.org/10.2139/ssrn.953888.

[19] Chen, X., Fan, Y. Estimation of copula-based semiparametric time series models. Journal of Econometrics, 2006.

[20] Data Science Genie. COMPUTING THE PORTFOLIO VAR USING COPULAS [online]. Data Science Genie, 2024 [viewed 15 May 2024]. Available from: https://datasciencegenie.com/computing-the-portfolio-var-using-copulas/

[21] Enders, W. Applied Econometric Time Series, Fourth edition. Publisher: Wiley; October 22, 2014.

[22] Engle, R., Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. Econometrica. Vol. 50, pp. 987–1008, 1982.

[23] Fantazzini, D. Dynamic Copula Modelling for Value at Risk. Frontiers in Finance and Economics, 2008 – Vol. 5, Issue 2. – P. 72-108.

[24] Francis, J.C., Kim, D. Modern Portfolio Theory: Foundations, Analysis, and New Developments. Published by John Wiley & Sons, Inc., Hoboken, New Jersey, 2013.

[25] Frees, E.W., Valdez, E.A. Understanding relationships using copulas. North American Actuarial Journal. 1998, 2, 1–25. Available from: doi: 10.1080/10920277.1998.10595667.

[26] Ganti, A., Murry, C., Veslaquez, V. Efficient Frontier: What It Is and How Investors Use It.[online] Investopedia, 2023 [viewed 19 May 2024]. Available from: https://www.investopedia.com/terms/e/efficientfrontier.asp#:~:text=The%20efficient%20frontier%20is%20the,for%20the%20level%20of%20risk.

[27] Ghalanos, A. Introduction to the rugarch package [online]. RStudio, PBC, 2022 [viewed 15 May 2024]. Available from: https://cran.rproject.org/web/packages/rugarch/vignettes/Introduction_to_the_rugarch_package.pdf

[28] Haugh, M. Mean-Variance Optimization and the CAPM. [online] IEOR E4706: Foundations of Financial Engineering, 2016 [viewed 19 May 2024]. Available from: https://www.columbia.edu/~mh2078/FoundationsFE/MeanVariance-CAPM.pdf

[29] Krzemienowsk, A., Szymczyk, S. Portfolio optimization with a copula-based extension of conditional value-at-risk. [online] Springerlink.com [viewed 15 May 2024]. Available from: https://link.springer.com/article/10.1007/s10479-014-1625-3

[30] Nelsen, R. An Introduction to Copulas. Second Edition. New York: Springer, 2006. 249 p. ISBN 978-0387-28659-4.

[31] Rosenberg, J., Schuermann, T. A. General Approach to Integrated Risk Management with Skewed, Fat-Tailed Risks. Journal of Financial Economics, 2006. Vol. 79, pp. 569 – 614. Available from: doi: https://doi.org/10.1016/j.jfineco.2005.03.001

[32] Sahamkhadam, M., Stephan, A., Östermark, R. Copula-based Black–Litterman portfolio optimization. European Journal of Operational Research, 2022, Available from: doi: https://doi.org/10.1016/j.ejor.2021.06.015.

[33] Sahamkhadam, M., Stephan, A., Östermark, R. Portfolio optimization based on GARCH-EVT-Copula forecasting models. International Journal of Forecasting. 2018 – Vol. 34, Issue 3. – P. 497-506.

[34] Shchankina, A., Ping, Z. An Empirical Study on the Modelling of an Optimal Investment Portfolio Using Multivariate Model of Conditional Heteroscedasticity: Evidence from the Chinese Stock Exchanges. In: International Scientific Conference "Far East Con" (ISCFEC 2020). China. Atlantis Press SARL, 2020. Available from: doi: 10.2991/aebmr.k.200312.179

[35] Sklar, A. Fonctions de r'epartition 'a n dimensionset leurs marges. Publ. Inst. Statis. Univ. 1959. Paris 8, pp.229–231.

[36] Statista. Largest stock exchanges in Europe, Middle East, and Africa (EMEA) as of March 2024, by market capitalization of listed companies. [online]. Statista, 2024 [viewed 15 May 2024]. Available from: https://www.statista.com/statistics/265251/domestic-market-capitalization-in-europe-the-middle-east-and-africa/

[37] Tang, A., Valdez, E. Economic Capital and the Aggregation of Risks using Copulas. Sydney: University of New South Wales (UNSW) and University of Connecticut, 2006. Available from: doi: http://dx.doi.org/10.2139/ssrn.1347675

[38] Tuan Anh, L., Thi Thanh Binh, D. Portfolio optimization under mean CVaR simulation with copulas on the Vietnamese stock exchange. Investment Management and Financial Innovations. 2021, 18(2), 273-286. Available from: doi:10.21511/imfi.18(2).2021.22.

[39] Zhao, Z., Zhou, R., Palomar, D. P., Feng, Y. Portfolio Optimization, 2019 IEEE Signal Processing Magazine, 2019.