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Application of the Method of Complete Bifurcation Groups in Parametrically Excited Pendulum Systems

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***Abstract* - An application of the new method of complete bifurcation groups (MCBG) in parametrically excited pendulum systems with an additional linear restoring moment and with the periodically vibrating point of suspension in both directions is introduced. Behaviour of the driven damped pendulum systems may be complex and with unexpected phenomena. Recent efforts in nonlinear dynamics show, that rare attractors (RA) have been found in all typical nonlinear models. Construction of complete bifurcation groups is based on the method of stable and unstable periodic regimes continuation on a parameter. This method is based on the ideas of Poincaré, Birkhoff, Andronov and others [6]. Global bifurcation analysis of the parametric pendulum systems allows to find new bifurcation groups, rare attractors and chaotic regimes. All results were obtained numerically, using our software.**

***Keywords* - complete bifurcation analysis, pendulum system, vibrating point of suspension, method of complete bifurcation groups, rare attractors, chaos, domains of attraction.**

I. INTRODUCTION

Behaviour of the driven damped pendulum systems may be complex and with unexpected phenomena [1-3]. Recent efforts in nonlinear dynamics show, that rare attractors (RA) have been found in all typical nonlinear models [4-7]. The systematic research of rare attractors is based on the method of complete bifurcation groups (MCBG), which allows conducting more complete global analysis of the dynamical systems. The main idea of the approach is periodic branch continuation along stable and unstable solutions. The method is based on the ideas of Poincaré, Birkhoff, Andronov and others [6]. It is shown that the MCBG allows to find important unknown attractors and new bifurcation groups in different nonlinear models. The main features of this method are illustrated in this work by a parametrically excited pendulum system with one degree of freedom.

In this paper authors apply the MCBG to global bifurcation analysis of a pendulum system with an additional linear restoring moment and with the periodically vibrating point of suspension in both directions. Rare attractors have been found building the complete bifurcation diagrams with stable and unstable periodic solutions. Among complete bifurcation periodic groups there are groups with rare periodic and rare chaotic attractors. All results were obtained numerically, using software NLO and SPRING, created in Riga Technical University [3, 4].

II. A DYNAMICAL PARAMETRICALLY EXCITED PENDULUM MODEL

The studied dynamical model is shown in Fig.1a. The system has additional linear restoring moment with the harmonically vibrating point of suspension in both directions. The system has three equilibrium positions (Fig.1d). Backbone curves and restoring moment for the system are shown in Fig.1b,1c. Close models have been examined in some works [8-16]. The aim of our bifurcation analysis is to find new unknown attractors and new bifurcation groups.

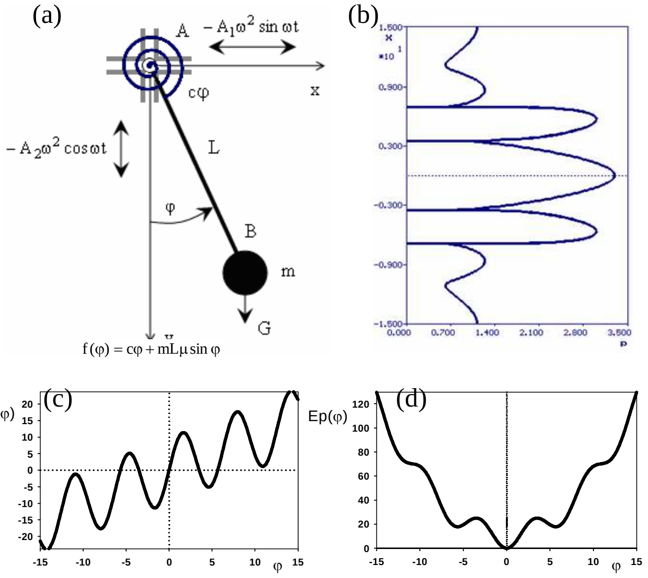

Fig. 1. The parametrically excited pendulum system with an additional linear restoring moment and with the periodically vibrating point of suspension in both directions. (a) Physical model; (b) backbone curve; (c) restoring moment; (d) potential well

The equation of motion for pendulum (Fig.1a) is such:

$$mL^2\ddot{\varphi} + b\dot{\varphi} + c\varphi + mL\left(\mu - A_2\omega^2 \cos\omega t\right)\sin\varphi + \\ + mLA_1\omega^2 \sin\omega t \cos\varphi = 0 \qquad (1)$$

where φ – angle, read-out from a vertical line; $\dot{\varphi}$ - angular velocity of the pendulum, where $\dot{\varphi} = d\varphi / dt$; t – time; m – mass, L – length of the pendulum; μ – gravitation constant; b – damping coefficient; c – linear stiffness coefficient; A_1, A_2 *and* ω – oscillation amplitudes and frequency of the point of suspension on a horizontal and a vertical direction.

Equation (1) will be used for bifurcation analysis of the pendulum systems.

III. RESULTS

The results of bifurcation analysis of the model (1) are represented in Fig. 2-15. In the first special case the model only with horizontal external force has three simple 1T bifurcation groups (Fig. 2).

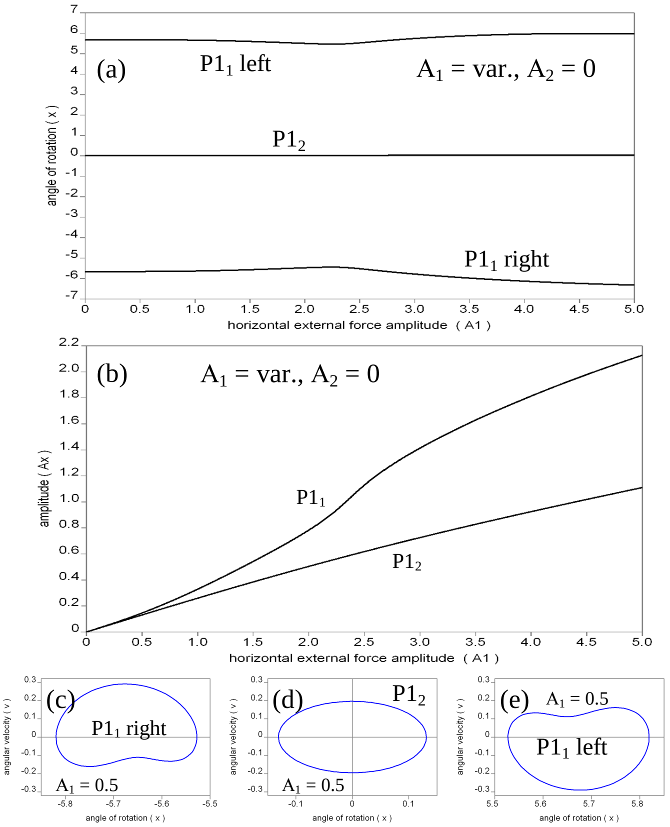

Fig. 2. Parametrically excited pendulum system (see Fig. 1a) with linear restoring moment and with the periodically vibrating point of suspension in horizontal direction. (a), (b) Bifurcation diagrams: state (φ, Amplitude) of the fixed periodic points versus horizontal external force amplitude A1. There are three simple 1T bifurcation groups. (c), (d), (e) Phase projections for cross-section A1 = 0.5. Parameters: m =1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = var., A2 = 0

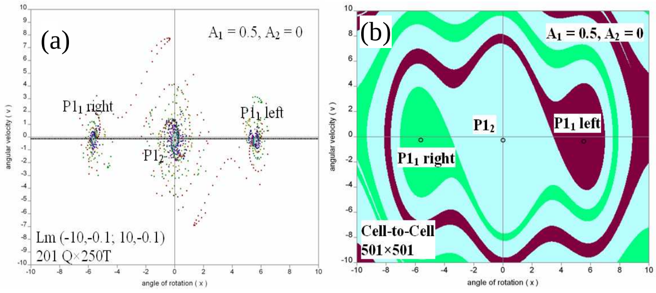

Fig. 3. Parametrically excited pendulum system (see Fig. 1a) with linear restoring moment and with the periodically vibrating point of suspension in horizontal direction. (a) Dynamical wells on the Poincaré map built by Lm(-10, -0.1; 10,-0.1) 201Q×250T. (b) Attractor-basin phase portrait with 501×501 grid of initial conditions for Eq.(1) of Fig. 2a. Parameters: m=1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0.5, A2 = 0

Phase projections for cross-section A_1 = 0.5 are shown in Fig. 2c, 2d, 2e. Dynamical wells built by Line mapping from a line (-10,-0.1; 10,-0.1) on the Poincaré map are represented on Fig. 3a. Using Cell-to-Cell mapping with 501×501 grid of initial conditions, domains of attraction (Fig. 3b) have been obtained for cross-section A_1 = 0.5 (see Fig. 2a).

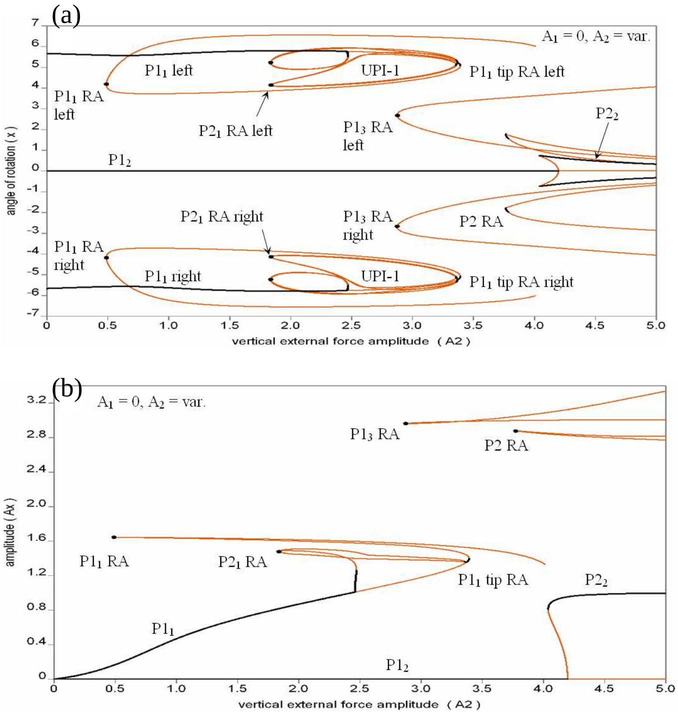

Fig. 4. Parametrically excited pendulum system (see Fig. 1a) with linear restoring moment and with the periodically vibrating point of suspension in vertical direction. (a), (b) Bifurcation diagrams: state (φ, Amplitude) of the fixed periodic points versus vertical external force amplitude A2. There are three symmetric 1T and one 2T bifurcation groups. Parameters: m = 1, L= 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0, A2 = var.

In the second special case the model only with vertical external force have three symmetric 1T and one 2T bifurcation groups (Fig. 4). In this figure stable solutions are plotted by solid lines and unstable – by thin lines (reddish online). In Fig. 5 coexistence of P1$_1$ twins stable solutions and P1$_1$ RA twins rare attractors (see Fig. 4) in cross-section A_2 = 0.4882 is shown. Domains of attractions obtained using two different methods are shown in Fig. 6: (a) insets and outsets from two symmetric 1T saddles; (b) Cell-to-Cell mapping with 501×501 grid of initial conditions.

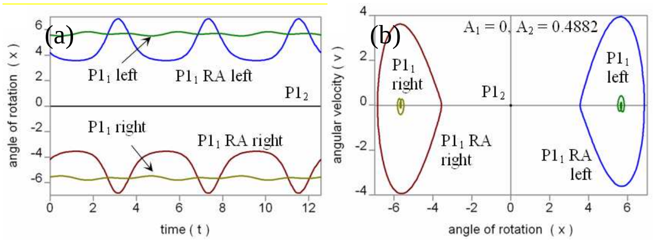

Fig. 5. Coexistence of P1$_1$ twins stable solutions and P1$_1$ RA twins rare attractors (see Fig. 4) in cross-section A_2 = 0.4882. (a) Time histories, (b) phase projections. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0, A2 = 0.4882

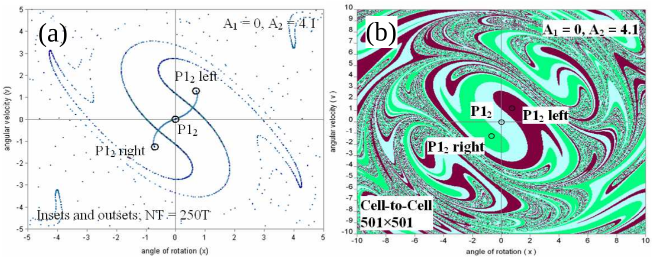

Fig. 6. Parametrically excited pendulum system (see Fig. 1a) with linear restoring moment and with the periodically vibrating point of suspension in vertical direction. (a) Insets and outsets from two symmetric 1T saddles (±0.586055, ±0.516811; ρ1 = 1.112, ρ2 = 0.389). (b) Attractor-basin phase portrait with 501×501 grid of initial conditions for Eq.(1) of Fig. 4a. Parameters: m=1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0, A2 = 4.1

Five different 1T bifurcation groups and one 2T bifurcation group have been found (Fig. 7) in general case for parametrically excited pendulum system with linear restoring moment and with the periodically vibrating point of suspension in both directions. Two of these groups are topologically similar and have rare attractors of a tip kind P1$_1$ RA and P1$_3$ RA. Two period one brunches near A_2 = 4 are not completed, because of problems of singularity. Other three 1T bifurcation groups have the own rare attractors P1$_4$ RA and P1$_5$ RA, which are stable in small parameter regions. Some cross-sections (A_2 = const) of bifurcation diagrams with dynamical characteristics from Fig. 7 are represented in Fig. 8, 9, 13, 14, 15. All attractors are of tip kind so each of them has not only periodic attractors, but also chaotic attractors as well.

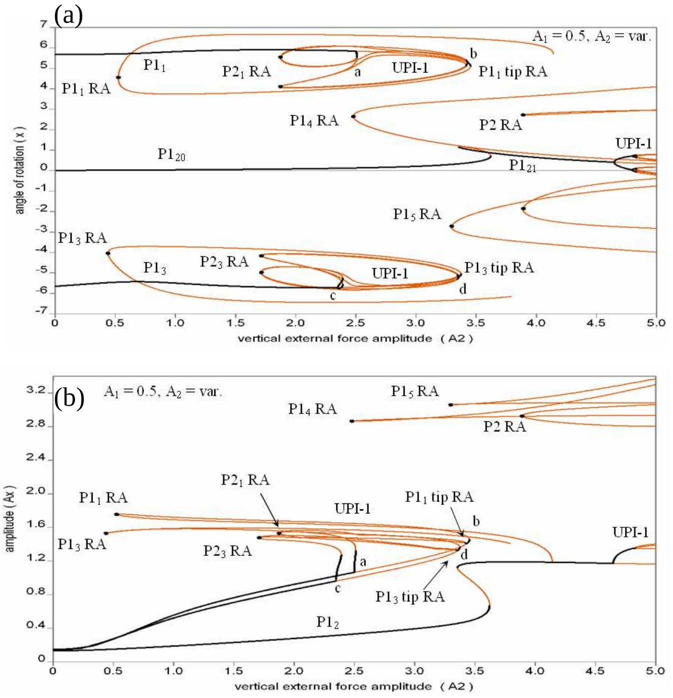

Fig. 7. The parametrically excited pendulum system (Eq. 1) with linear restoring moment and with the periodically vibrating point of suspension in both directions. (a), (b) Bifurcation diagrams: state (φ, Amplitude) of the fixed periodic points versus vertical external force amplitude A2. There are five 1T and one 2T bifurcation groups. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0.5, A2 = var.

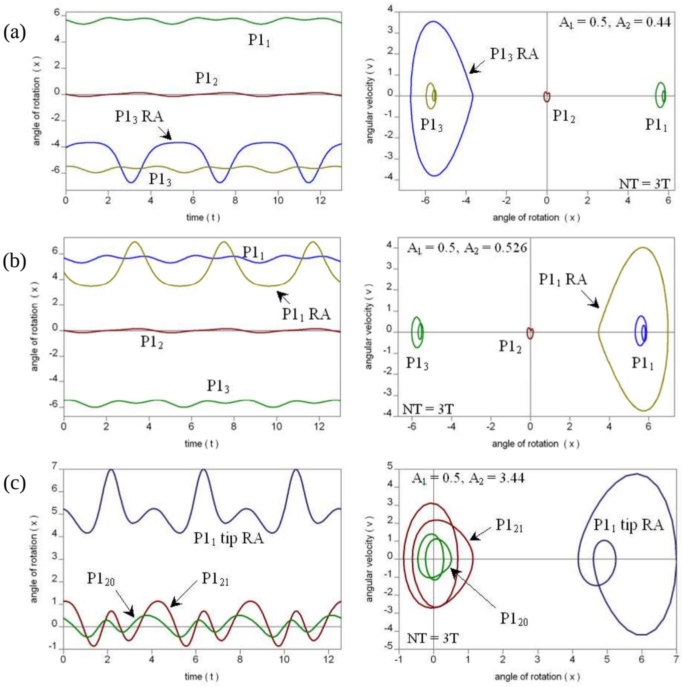

Fig. 8. Coexistence of P1 usual and rare attractors P1 RA for three cross-sections (see Fig. 7). (a) Time histories and phase projections for A2 = 0.44. The rare attractor P13 RA has the fixed point FP (-4.05606 /1.17632). (b) The same for A2 = 0.526. Rare attractor P11 RA has the FP (4.56968/-2.60245). (c) The same for A2 = 3.44. Rare attractor P11 tip RA has the FP (5.23616/-0.315143). Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0.5, A2 = var.

The examples of coexistence of period-1 (P1) stable solutions and P1 RA rare attractors for three cross-sections A2 = 0.44, A2 = 0.526 and A2 = 3.44 on bifurcation diagram (Fig. 7a) with the time histories and phases projections are shown in Fig. 8. Oscillation amplitudes of rare attractors in some cases are tenfold bigger than oscillating amplitudes of stable P1 regimes. The examined system has also other bifurcation group of higher order with rare attractors, for example, 2T bifurcation group with P2 RA rare attractor with large oscillation amplitudes.

Each bifurcation group has its own unstable periodic infinitium (UPI) [4-7] with corresponding chaotic attractors. The example of globally stable chaotic attractor for cross-section with parameters A_1 = 0.5, A_2 = 4.9, obtained using the contour mapping [4-7], is shown in Fig. 9. The system also has other nT subharmonic bifurcation groups with n = 3, not shown.

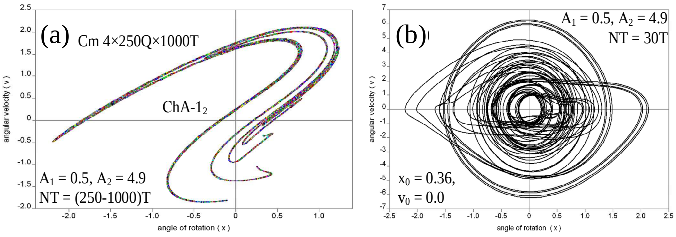

Fig. 9. Chaotic attractor in the parametrically excited pendulum system for cross-section A2 = 4.9 of bifurcation diagram Fig. 7: (a) Poincaré map - Cm 4×250Q×(250-1000)T; (b) phase projection with NT=30T. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ=9.81, ω = 1.5, A1 = 0.5, A2 = 4.9

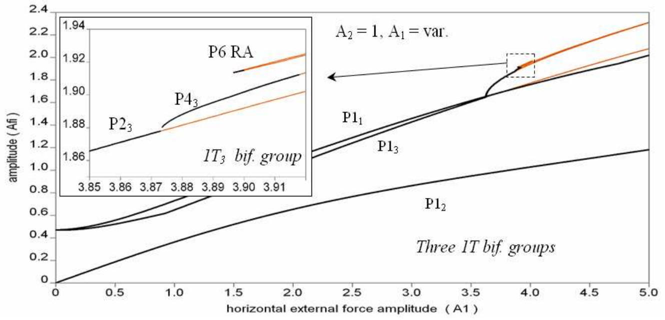

Fig. 10. The parametrically excited pendulum system (Eq. 1) with linear restoring moment and with the periodically vibrating point of suspension in both directions. Bifurcation diagram - state (Amplitude) of the fixed periodic points versus horizontal external force amplitude A1. There are three 1T and one 6T bifurcation groups. Parameters: m = 1, L = 1, b = 0.2, c= 1, μ = 9.81, ω = 1.5, A1 = var., A2 = 1

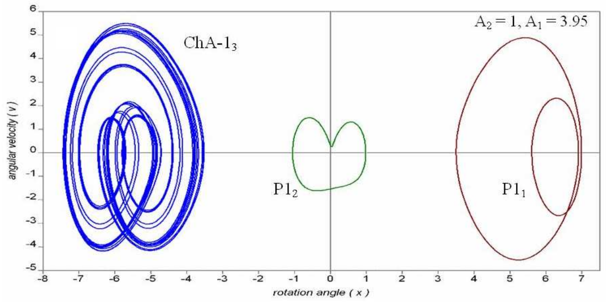

Fig. 11. Phase projections for coexisting periodic regimes P11, P12 and ChA-13 chaotic attractor for cross-section A1 = 3.95 (see Fig.10). Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 3.95, A2 = 1

Coexistence of P1 stable solutions and ChA-1$_3$ chaotic attractor is shown on phase projections (Fig. 11) and on Poincaré map - Cm 4×250Q×1000T (Fig. 12).At cross-section A1 = 3.95 of bifurcation diagram (see Fig. 10) in the parametrically excited pendulum system with additional linear restoring moment and vibrating point of suspension in both directions there are periodic regimes P11, P12 and chaotic attractor ChA-13. Domains of attraction of periodic solutions in parametrically excited pendulum system for different cross-sections of bifurcation diagram (see Fig. 7) obtained by Cell-

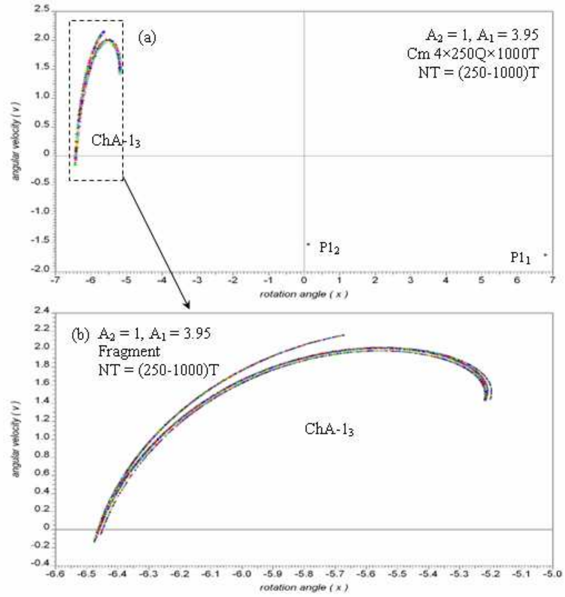

Fig. 12. Coexistence of P1 stable solutions and ChA-13 chaotic attractor for cross-section A1 = 3.95 in the parametrically excited pendulum system with additional linear restoring moment (see Fig. 10): a) Poincaré map - Cm 4×250Q×1000T; b) fragment of contour mapping. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 3.95, A2 = 1

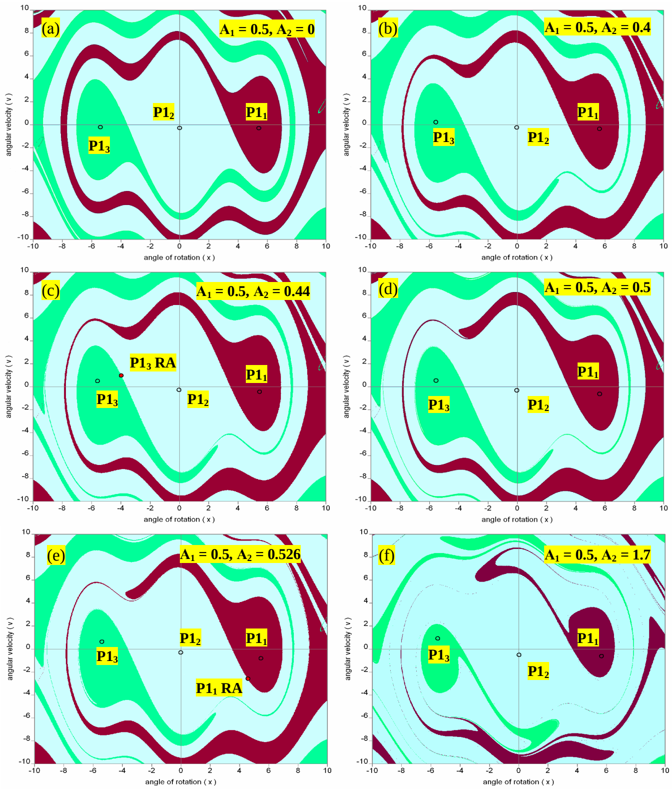

Fig. 13. Attractor-basin phase portraits with 501×501 grid of initial conditions for Eq.(1) of Fig. 7a. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0.5, A2 = var.

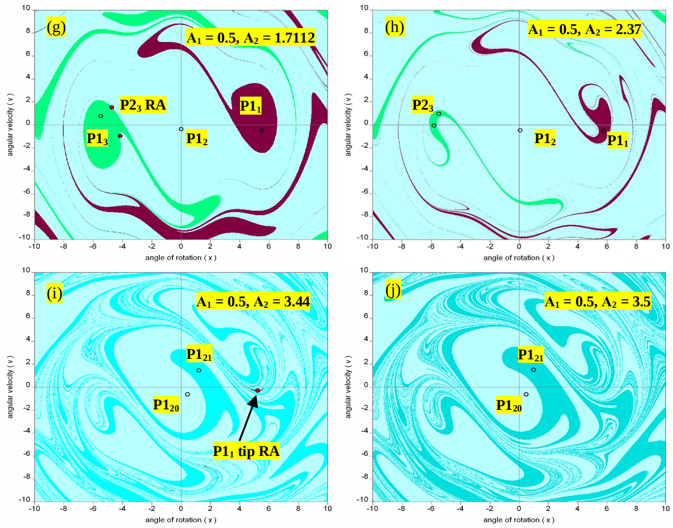

Fig. 14. Attractor-basin phase portraits with 501×501 grid of initial conditions for Eq.(1) of Fig. 7a. Parameters: m = 1, L = 1, b = 0.2, c = 1, μ = 9.81, ω = 1.5, A1 = 0.5, A2 = var.

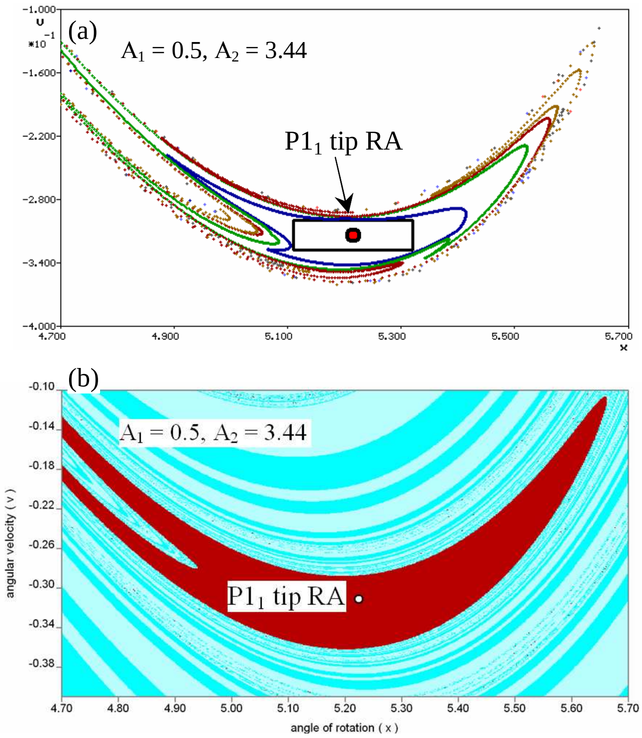

Fig. 15. Domains of attraction of period-1 tip rare attractor P11 tip RA (see Fig. 7a and 14i) in a parametrically excited pendulum system: (a) obtained by using a reverse contour mapping from a rectangle; (b) obtained by using Cell-to-Cell mapping with 501×501 grid of initial conditions. Parameters: m = 1, L= 1, b = 0.2, c = 1, µ = 9.81, ω = 1.5, A1 = 0.5, A2 = 3.44

-to-Cell mapping, are shown in Fig. 13-14.Comparison of two different methods for building domains of attraction of tip kind rare attractor $P1_1$ RA is represented in Fig. 15: (a) reverse contour mapping from a rectangle; (b) Cell-to-Cell mapping with 501×501 grid of initial conditions. These methods show the same results of building domains of attraction.

IV. CONCLUSIONS

It is shown that using of the method of complete bifurcation groups allows to conduct the bifurcation analysis of a parametrically excited pendulum systems with additional linear, and to find new bifurcation groups, rare attractors and chaotic regimes. Amplitudes of rare attractors may be grater than once of regular attractors. It might be supposed that rare attractors can result in the loss of control and stability of oscillating systems, or in the other catastrophic or unexpected phenomena.

Some obtained new effects can be used for the parametric stabilization of unstable oscillations in technological processes. Authors hope to attract attention of scientists and research engineers to the important problem of nonlinear oscillations analysis and search of rare attractors in the pendulum-like or other dynamical systems.

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Aleksejs Klokovs. Pilnu bifurkācijas grupu metodes pielietojums parametriski ierosinātās svārstu sistēmās
Tiek aplūkots jaunas pilnu bifurkācijas grupu metodes (MPBG) pielietojums parametriski ierosinātās svārstu sistēmās ar papildus lineāru atjaunotājmomentu un ar periodiski vibrējošu piekares punktu divos virzienos. Ierosinātu disipatīvu svārstu sistēmu uzvedība var būt sarežģīta un ar negaidītiem fenomeniem. Neseni pētījumi nelineārā dinamikā rāda, ka retie atraktori (RA) tika atrasti visos tipiskos nelineāros modeļos. Pilnu bifurkācijas grupu uzbūves metodes pamatā ir stabilu un nestabilu periodisku režīmu turpinājums. Šī metode ir izveidota, pamatojoties uz Puankarē, Birkhofa, Andronova un citu [6] idejām. Parametrisku svārstu sistēmu globāla bifurkācijas analīze ļauj atrast jaunas bifurkācijas grupas, retus atraktorus un haotiskus režīmus.

Алексей Клоков. Применение метода полных бифуркационных групп в параметрически возмущенных маятниковых системах
Представлено приложение нового метода полных бифуркационных групп (МПБГ) в параметрически возмущенных маятниковых системах с дополнительным линейным восстанавливающим моментом и с периодически вибрирующей точкой подвеса в двух направлениях. Поведение вынужденных диссипативных маятниковых систем может быть сложным и с неожиданными феноменами. Недавние исследования в нелинейной динамике показывают, что редкие аттракторы (РА) были найдены во всех типичных нелинейных моделях. Построение полных бифуркационных групп основано на методе продолжения устойчивых и неустойчивых периодических режимов по параметру. Этот метод основан на идеях Пуанкаре́, Биргофа, Андронова и других [6]. Глобальный бифуркационный анализ параметрических маятниковых систем позволяет находить новые бифуркационные группы, редкие аттракторы и хаотические режимы.