RIGA TECHNICAL UNIVERSITY
Faculty of Transport and Mechanical Engineering
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"Transport"

OPTIMAL LOCATION OF SERVICE OBJECTS WITHIN THE EXISTING TRANSPORT INFRASTRUCTURE

Doctoral thesis summary

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DOCTORAL THESIS
IS PROPOSED FOR ACHIEVING
SCIENTIFIC DEGREE OF DOCTOR OF SCIENCE
IN ENGINEERING AT RIGA TECHNICAL UNIVERSITY

The defence of the doctoral thesis for scientific degree of Doctor of Science in Engineering will take place at an open session on December 6, 2011, at 14:30, of the Riga Technical University, FTME, Institute of Transport Vehicle Technologies, Lomonosova street 1, building V, room 218, Riga, Latvia.

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CONFIRMATION

Hereby I confirm that I have worked out the present doctoral thesis, which is submitted for consideration at Riga Technical University for achieving scientific degree of Doctor of Science in Engineering. This doctoral thesis is not submitted to any other university for achieving scientific degree.

Andrey Kashurin……………………………. (Signature)

Date: November 1, 2011

The doctoral thesis is written in English. The doctoral thesis consists of 8 chapters. The references include 81 sources. There are 45 figures, 77 formulas and 30 tables to illustrate the conception of the carried out research. The doctoral thesis contains 161 pages.

ABSTRACT

The doctoral thesis «Optimal location of service objects within the existing transport infrastructure» has been worked out by Andrey Kashurin to get the scientific degree of Doctor of Science in Engineering. Scientific supervisor of the work is Dr. habil. sc. ing., professor Alexander Andronov.

The goal of the doctoral thesis is development of the mathematical models, methods, algorithms and computer programs for the problem of optimal location of service objects within the existing transport infrastructure, investigation of the elaborated methods efficiency and decision of the real actual problem. Optimal location of a service object is practically important problem. Possible object types are technical maintenance stations, medical aid, sports facilities and others. It is necessary solve this problem using various factors, and first of all an existing infrastructure.

In this connection the following tasks are considered:

- Analytical description of a distribution of population on the territory of Latvia, allowing for the specified coordinate of territory to calculate corresponding population's density.
- Development of the mathematical model for problem of optimal spatial location of service objects. Modern methods of the optimization theory are used.
- Development of special computer programs in Mathcad and MATLAB environment for the decision of the considered problems.
- The experimental research of efficiency of the developed methods and programs. Advantage of genetic algorithm in comparison with the algorithms based on a method of a gradient, for multiextreme problems, is shown. It is established that the best results are given by a combination of genetic algorithm and a gradient method.
- Decision of the real actual problem of service objects location, based on developed models, algorithms and methods by the instrumentality of developed computer programs.

CONTENTS

1. INTRODUCTION

Present doctoral thesis «Optimal location of service objects within the existing transport infrastructure» is devoted to problem of service object location in spatial space, on the basis of use of the modern mathematical methods like probability theory, nonlinear optimization and genetic algorithms.

1.1. Actuality of the problem

Service object location is and has been a well established research area within Operations Research (OR). Numerous papers and books are witnesses of this fact [18]. The American Mathematical Society (AMS) even created specific codes for location problems (90B80 for discrete location and assignment, and 90B85 for continuous location). Nevertheless, the question of the applicability of location models has always been under discussion [38].

A general service object location problem involves a set of spatially distributed customers and a set of service objects to serve customer demands [18], [52]. The questions to be answered are:

• Which service object should be used (opened)?

• Which customers should be serviced from which service object (or service objects) so as to minimize the total costs [38]?

Service objects location, also known as location analysis, is a branch of operations research concerning itself with mathematical modeling and solution of problems concerning optimal placement of service objects in order to minimize transportation costs, avoid placing hazardous materials near housing, outperform competitors' facilities, etc. Location models are used in a variety of applications such as locating warehouses within a supply chain to minimize the average time to market, locating noxious material to maximize its distance to the public, etc.

Optimal location of service objects is practically very important problem. Main aim is minimizing the cost of satisfying some set of demands (of the customers) with respect to some set of constraints. Service objects location decisions are critical elements in strategic planning for a wide range of private and public firms [77].

The basic task of all variants of service object location problems is the following: a company wants to open up a number of service objects to serve their customers. Both the opening of a service object at a specific area or the service of a particular customer through a service object incurs some cost. The goal is to minimize the overall cost associated to a specific way of opening up service objects and serving customers [22].

Service objects are objects of the same class complying with the following requirements:

- All objects are of the same type, i.e. they perform the same manufacturing or transport tasks.
- All customers using the relevant services can, in principle, use a service of any object.

It is possible to present the following examples:

- Service stations
- Filling stations
- Sports bases, etc.

In our work a problem of service object location on the territory of Latvia is considered. It is necessary to place the predefined number of objects, to minimize loss function (transport expenses). This problem is solved using a density function of serviced object location and an accepted loss function.

1.2. Objectives and tasks of the research

The *objective* of the doctoral thesis is:

Development of the mathematical models, methods and computer programme package for the problem of optimal location of service objects.

In this connection the following *tasks* are considered:

1. Consideration of the existing methods and models of optimal location of the service objects.
2. Statistical description of a distribution of population's density on the territory of Latvia.
3. Design of the research information data base, making statistical data collections for the regions of Latvia.
4. Working out the analytical description of a distribution of population on the territory of Latvia, allowing for the specified coordinate of territory to calculate corresponding population's density.
5. Development of the mathematical model for problem of optimal spatial location of service objects.
6. Description of principles of the numerical optimization and survey some exciting numerical methods and algorithms of an optimization.
7. Development of the optimization methods for elaborated mathematical model.
8. Investigation of the elaborated optimization methods' efficiency.

9. Development of the special computer programs for the suggested optimization methods.
10. Consideration of the real actual problem of service objects location on the territory of Latvia, based on developed models, methods, algorithms and computer programs.

1.3. Methodology and methods of the research

The doctoral thesis research is based on:

1. Modern optimization methods and algorithms such a line search methods (step length, the Wolfe conditions, backtracking line search), gradient optimization, quasi-Newton methods (the BFGS method), and genetic algorithm.
2. Methods of probability theory and mathematical statistics.
3. Statistical data received from "*Central Statistical Bureau of Latvia*" (*LR Centrālā statistikas pārvalde, LR CSP*) and Ministry of Education and Science of the Republic of Latvia data base *"Sports facilities register*", which one was created in collaboration with the author of the doctoral thesis.
4. Scientific literature, press releases and Internet-sources devoted to the investigated problems.
5. Computer based support for necessary calculation and investigation, i.e. PTC Mathcad 14, MathWorks MATLAB R2009b environment.

1.4. Scientific novelty

Novelty of the present research consists of:

1. Analytical descriptions of a population distribution on the territory of Latvia, allowing for the specified coordinate of territory calculate corresponding population's density.
2. The original mathematical model for optimal spatial location of service objects.
3. Improved optimization method, used a combination of the genetic algorithm and gradient method.
4. Package of the special computer programs for the problem of optimal spatial location of service objects.

1.5. Practical importance and applying

1. On the basis of the obtained results a part of lectures and practical works on the study discipline "Computer Methods for Engineering Problem Solving" for the first year student of master's studies programme of the Riga Technical University Institute of Transport Vehicle Technologies is prepared.
2. Models and methods of the description of a distribution of population on the territory of Latvia, which are developed by the author, were used in the scientific project "Creation of mathematical models, algorithms and computer programs for Latvia's transport system's analysis, development prognosis and optimization", which was a component of the Scientific Project "Zinātniskās darbības attīstība augstskolās" and lasted from June 1 till December 31, 2008.
3. According to the Article 12 of the Law on Sports [63] titled "Sports Facilities", The National Sports Development Program for 2006-2012 (approved by Cabinet of Ministers Instruction No. 838, October 31, 2006) and The Cabinet of Ministers (Protocol No. 50 28.§) (this policy is in line with Latvia's National Development Plan (Cabinet of Ministers Instruction No. 564, July 4, 2006)) the problem of the optimal location of the new track and field arenas on the territory of Latvia is solved in collaboration with Sports Department of the Ministry of Education and Science of the Republic of Latvia.
4. The obtained models and algorithms can be used by private and government companies for the optimal location of the service objects, i.e. car technical maintenance stations, sports facilities, filling stations and others.

1.6. Structure of the doctoral thesis

Chapter 1. Methods and models of optimal location of the service objects. Problem of optimal location of service objects is described. Various approaches to a setting of the distance functions are considered. The lasts play an important role in the problem of interest. Various mathematical model and methods of location-allocation problem were analyzed.

Chapter 2. Statistical description of a distribution of population's density on the territory of Latvia. This Chapter is dedicated to analysis of the statistical data about population distribution on territory of Latvia. As result an analytical expression for population density has been obtained. One allows us for the specified coordinate of territory to calculate corresponding population's density

On this basis the following problem is solved: a weighted average centre of Latvia is found, i.e. coordinate, for which the total distance from all points of Latvia (subject to density) is minimal one. Necessary statistical information has been received from *Central Statistical Bureau of Latvia.*

Chapter 3. Survey of exciting numerical methods of an optimization. In this Chapter principles of the numerical optimization and some exciting numerical methods are discussed. The used in our work methods and algorithms of nonlinear optimization are described: the line search, the Wolfe conditions, the backtracking line etc.

Chapter 4. Mathematical model for problem of spatial location of service objects. A considered problem is stated. It is known a density function of serviced object location and the distance functions. A criterion of the location efficiency is average total loss, i.e. transport expenses. It is necessary to place the predefined number of objects, to minimize loss function. One-dimensional and two-dimensional cases of the problem are considered.

Chapter 5. Gradient and quasi-Newton algorithms of an optimization. Both algorithms are used for an optimal location of service object in spatial space. An elaborated search procedure satisfies the Wolfe conditions. Location of 4 service objects across territory of Latvia has been taken an example.

Chapter 6. Genetic algorithm. This Chapter is devoted to the function optimization using evolutionary algorithms such a genetic algorithms. Canonical genetic algorithm is described. Application of the genetic algorithm to the problem of optimal spatial location of service objects is described. To evaluate algorithm efficiency, plan of the experiments is designed. Numerical experiments are performed. The best results turn out by a combination of the genetic algorithm and gradient method.

Chapter 7. Description of programs. Computer programs for the optimal location of service objects are described. Two main programs are developed. First program (gradient and quasi-Newton optimization) written in Mathcad. Second (genetic algorithm optimization) written in MATLAB. Program structure, functions, variables and syntaxes are considered.

Chapter 8. Application of the considered methods. An actual problem of track and field arenas location on spatial space is considered. It is known a density function of serviced object location and a loss function. It is necessary to place the predefined number of objects, to minimize loss function (transport expenses). List and coordinates of the existing track and field arenas is taken form the Ministry of Education and Science of the Republic of Latvia data base "*Sports facilities register*" (http://sportabazes.izm.gov.lv/sbdb/). The received results have been approved by the Ministry of Education and Science of the Republic of Latvia.

1.7. Defensible theses

Author of the doctoral thesis defends:

1. Development of the mathematical models for the problem of optimal location of service stations.
2. Elaboration of the optimization methods and algorithms for the suggested mathematical model.
3. Elaboration of computer programme package for the model and method realization.
4. Investigation of the elaborated optimization methods’ efficiency.

1.8. Approbation of the doctoral thesis

The main results of the doctoral thesis have been presented on 9 international scientific conferences:

1. RTU 48th International Scientific Conference, October 11-13, 2007, Riga, Latvia
2. RTU 49th. International Scientific Conference, Information technology and management science, October 13-15, 2008, Riga, Latvia
3. RTU 49th International Scientific Conference, Mathematical methods of transportation systems control, October 13-15, 2008, Riga, Latvia
4. Innovation and new technologies conference, January 20-21, 2009, Riga, Latvia
5. RTU 50th International Scientific Conference, October 12-16, 2009, Riga, Latvia
6. International Conference on Accelerated Life Testing, Reliability-based Analysis and Design, May 19-21, 2010, Clermont-Ferrand, France
7. RTU 51st International Scientific Conference, October 10-15, 2010, Riga, Latvia
8. Applied Stochastic Models and Data Analysis (ASMDA2011), The 14th Conference of the ASMDA International, June 7-10, 2011, Rome, Italy
9. 8th International Scientific and Practical Conference “Environment. Technology. Resources.” June 20-22, 2011, Rezekne, Latvia

The author of the doctoral thesis is the author and co-author of 8 scientific research publications.

List of publications:

1. Andronovs A., Kashurin A. On a problem of spatial arrangement of service stations // Computer modelling and new technologies. - 2007, Vol.11, N. (2007) pp. 31-37.
2. Kashurin A. Statistical description of a distribution of population density over the Latvian territory // Scientific journal of RTU. 5th series. Datorzinātne. - Vol. 36 (2008), pp. 108-115.
3. Kashurin A. Problem of optimal spatial arrangement of service stations // Third international conference on accelerated life testing, Reliability-based analysis and design, France, Clermont-Ferrand, May 18-21, 2010. – pp. 249-254.
4. Kashurin A. A problem of arrangement of service stations on the given territory // Scientific journal of RTU. 6th series., Mašīnzinātne un transports. - 34. vol. (2010), pp. 111-116.
5. Kashurin A., Parkova I. Genetic algorithm of optimal spatial arrangement of service stations // Scientific proceedings of the 14th conference of the ASMDA International Society, Italy, Rome, June 7-10, 2011. pp. 667.
6. Parkova I., Kashurin A., Valishevsky A., Vilumsone A. Making decisions on arrangement of electronics in smart garment // Proceedings of the 8th International Scientific and Practical Conference “Environment. Technology. Resources.”, Latvia, Rezekne, 20.-22. June, 2011. pp. 202-211.
7. Parkova I., Valishevsky A., Kashurin A., Vilumsone A. Integration of flexible keypad into clothing // Proceedings of the 8th International Scientific and Practical Conference “Environment. Technology. Resources.”, Latvia, Rezekne, 20.-22. June, 2011. pp. 173-181.
8. Kashurin A. Application of the problem of optimal location of service stations // Scientific journal of RTU. Mašīnzinātne un transports, 6th series - Riga: RTU, 2011. - vol.34. Accepted for publication.

2. SUMMARY OF RESEARCH RESULTS

2.1. Statistical description of a distribution of population's density on the territory of Latvia

A number of vital practical problems require the description of a distribution of population on the considered territory. In this connection we have set the task to describe the available data in an analytical form on the population of Latvia.

Our problem consists of working out the analytical dependences, allowing for the specified coordinate of territory of the Latvia to calculate corresponding population's density.

On this basis the following problem is solved: a weighted average centre of Latvia is found, i.e. coordinate, for which the total distance from all points of Latvia (subject to density) is minimum.

On the Tables 2.1 and 2.2 the data about the population and the square of the territory of Latvia on cities and districts is represented, as well as their coordinates. Before July 1, 2009 territory of Latvia was divided into 26 districts and 7 cities. All the statistical data provided to us by *Central Statistical Bureau of Latvia*. On the Figure 2.1 the map of population's density of Latvia is shown. Information source is Latvian Association of Local and Regional Governments (LPS, http://www.lps.lv). The different colors correspond with different allocation in the given region.

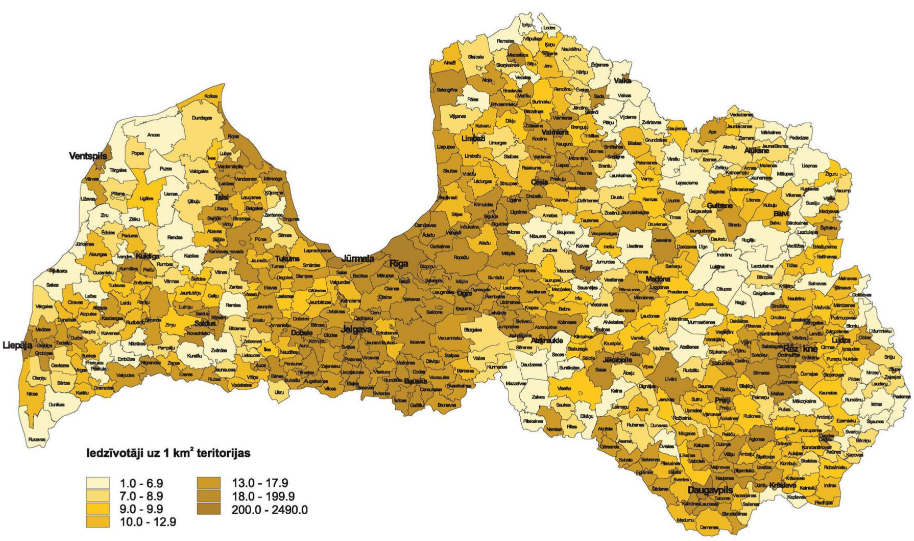

Figure 2.1 Map of population's density of Latvia

Table 2.1

Characteristics of big cities of Latvia

Index γ	City	Coordinate z_γ	Coordinate y_γ	Population $\hat{H}_\gamma$	Square P_γ km²
1	Rīga	190	150	722 485	307
2	Daugavpils	340	30	108 091	73
3	Jelgava	170	110	66 051	60
4	Jūrmala	150	180	55 408	100
5	Liepāja	10	100	85 477	60
6	Rēzekne	390	100	36 345	18
7	Ventspils	50	200	43 544	55

Table 2.2

Characteristics of districts of Latvia

Index ι	District	Coordinate z_ι	Coordinate y_ι	Population $\hat{H}_\iota$	Square P_ι km²
1	Aizkraukles district	270	110	40 116	2 567
2	Alūksnes district	360	210	24 483	2 245
3	Balvu district	390	170	27 245	2 381
4	Bauskas district	210	100	50 988	1 881
5	Cēsu district	270	180	56 614	2 973
6	Daugavpils district	340	40	39 496	2 526
7	Dobeles district	130	100	37 980	1 632
8	Gulbenes district	340	180	26 281	1 876
9	Jelgavas district	170	110	36 941	1 605
10	Jēkabpils district	300	90	52 593	2 997
11	Krāslavas district	400	50	33 313	2 288
12	Kuldīgas district	60	150	35 822	2 500
13	Liepājas district	30	110	43 849	3 593
14	Limbažu district	230	230	37 798	2 602
15	Ludzas district	420	110	31 305	2 412
16	Madonas district	330	140	42 918	3 349
17	Ogres district	240	130	64 060	1 843
18	Preiļu district	340	80	38 317	2 042
19	Rēzeknes district	380	100	40 442	2 809
20	Rīgas district	210	150	161119	3 132
21	Saldus district	50	110	36 735	2 182
22	Talsu district	100	200	46 680	2 748
23	Tukuma district	130	150	54 813	2 457
24	Valkas district	300	220	31 723	2 441
25	Valmieras district	260	240	58 328	2 373
26	Ventspils district	60	190	13 945	2 462

Reasonably the population of districts and big cities to be examined separately.

The districts population

Let $\imath$ be an district index, $\imath$ =1, 2, …, w, where w is a number of the considered districts. Let's use following designations:

$SR_\imath$ be the territory of the $\imath$-th district (it's point $(z,y)\epsilon\ SR_\imath$);

$\xi_\imath(z,y)$ be the indicator's function of the $\imath$-th district:

$$\xi_\imath(z,y)=\begin{cases} 1, \textit{ if } (z,y)\ \epsilon\ SR_\imath, \\ 0, \textit{ otherwise}, \end{cases}$$

$\hat{H}_\imath$ be population of the $\imath$-th district,

$P_\imath$ be square of the $\imath$-th district, km^2,

$h_\imath$ be population's density for the $\imath$-th district: $h_\imath = \hat{H}_\imath / P_\imath$.

We assume that every district is represented by a circle. Then the territory of Latvia is divided into circles, which characteristics are given in Table 2.4. The circle radius is calculated judging by a condition, that the square of the circle and district coincide. Let

$z_\imath, y_\imath$, be coordinates of the centre of the $\imath$-th district

$r_\imath$ be radius of the $\imath$-th district $r_\imath$, calculated by the formula

$$r_\imath = \sqrt{P_\imath / \pi}\ . \tag{2.1}$$

For the considered case function $\xi_\imath\ (z,y)$ is described as following

$$\xi_\imath\ (z,y)=\begin{cases} 1, \textit{ if } (y_\imath - r_\imath < y < y_\imath + r_\imath) \wedge \left\lfloor z_\imath - \sqrt{(r_\imath)^2 - (y-y_\imath)^2} < z < z_\imath + \sqrt{(r_\imath)^2 - (y-y_\imath)^2} \right\rfloor \\ 0, \textit{ otherwise}. \end{cases} \tag{2.2}$$

We suppose that objects are placed uniformly on the given territory. Therefore population density in point (z,y) is described as following function

$$f^{(r)}(z,y) = \sum_{i=1}^{w} h_\imath \xi_\imath (z,y)\ . \tag{2.3}$$

The population of big cities

Now we consider big cities.

Let γ be a city's index, γ=1, 2, …, m, where m is a number of considered big cities. Let's denote:

SP_γ be the territory of the γ-th city;

z_γ, y_γ, are coordinates of the centre of the γ-th city;

$\psi_\gamma(z,y)$ be an indicator's function of the γ-th city:

$$\psi_\gamma(z,y)= \begin{cases} 1, \; if \; (z,y) \in SP_\gamma, \\ 0, \; otherwise. \end{cases}$$

Indicator's function $\psi_\gamma(z,y)$ is described as following

$$\psi_\gamma(z,y)= \begin{cases} 1, \; if \; (y_\gamma - r_\gamma < y < y_\gamma + r_\gamma) \wedge \left[z_\gamma - \sqrt{(r_\gamma)^2 - (y - y_\gamma)^2} < z < z_\gamma + \sqrt{(r_\gamma)^2 - (y - y_\gamma)^2} \right] \\ 0, \; otherwise. \end{cases} \tag{2.4}$$

z_γ, y_γ, - coordinates of the centre of the γ-th city;

r_γ – radius of the γ–th city, calculated by the formula (2.1).

The populations densities of the cities calculated by formula $h_\gamma = \hat{H}_\gamma / P_\gamma$, where

$\hat{H}_\gamma$ – population of the γ-th city;

P_γ – is square of the γ-th city, km^2.

The density of distribution of the population of γ-th city on territory will be described by two-dimensional normal density:

$$f^{(p)}(z,y) = \frac{h_\gamma}{2\pi\sigma_\gamma^2} \exp\left\{ -\frac{1}{2\sigma_\gamma^2} \left[(z - z_\gamma)^2 + (y - y_\gamma)^2 \right] \right\}, \tag{2.5}$$

where σ_j is a standard deviation for the density of distribution of the population calculated by the formula:

$$\sigma_\gamma = \rho r_\gamma,, \tag{2.6}$$

where ρ>0,5 – is a coefficient of city's attraction.

General expression for population density in a point (z, y) will be

$$f(z,y) = f^{(r)}(z,y) + \sum_{\gamma=1}^{m} \psi_\gamma(z,y) f^{(p)}(z,y). \tag{2.7}$$

The main characteristics of the districts and the cities are presented in the Table 2.3 and Table 2.4.

Table 2.3

Population's density, radius and standard deviation for the cities

City	Density	Radius	σ_j
Rīga	2353,37	9,89	6,92
Daugavpils	1480,70	4,82	3,37
Jelgava	1100,85	4,37	3,06
Jūrmala	554,08	5,64	3,95
Liepāja	1424,62	4,37	3,06
Rēzekne	2019,17	2,39	1,68
Ventspils	791,71	4,18	2,93

Table 2.4

Population's density, radius of district

City / District	Density	Radius
Aizkraukles district	15,63	28,59
Alūksnes district	10,91	26,73
Balvu district	11,44	27,53
Bauskas district	27,11	24,47
Cēsu district	19,04	30,76
Daugavpils district	15,64	28,36
Dobeles district	23,27	22,79
Gulbenes district	14,01	24,44
Jelgavas district	23,02	22,60
Jēkabpils district	17,55	30,89
Krāslavas district	14,56	26,99
Kuldīgas district	14,33	28,21
Liepājas district	12,20	33,82
Limbažu district	14,53	28,78
Ludzas district	12,98	27,71
Madonas district	12,82	32,65
Ogres district	34,76	24,22
Preiļu district	18,76	25,50
Rēzeknes district	14,40	29,90
Rīgas district	51,44	31,57
Saldus district	16,84	26,35
Talsu district	16,99	29,58
Tukuma district	22,31	27,97
Valkas district	13,00	27,88
Valmieras district	24,58	27,48
Ventspils district	5,66	27,99

Weighted average coordinates of Latvia

The weighted average coordinate of Latvia is a point *(u, v)* for which the following condition is satisfied: if all inhabitants of Latvia have to gather in one point, that *(u, v)* is a point, for which the total distance passed by all, will be minimum.

For simplification we will consider all population of γ-th city concentrated in coordinate of its centre (z_{γ}, y_{γ}). This is a reasonable assumption, because the squares of cities are essentially less then squares of districts.

Now mathematically the problem is formed in such way: *to find a point (u, v) which minimizes objective function*

$$g(u,v)=\sum_{\iota=1}^{w}h_{\iota}\int_{y_\iota-r_\iota}^{y_\iota+r_\iota}\int_{z_\iota-\sqrt{r_\iota^2-(\omega-y_\iota)^2}}^{z_\iota+\sqrt{r_\iota^2-(\omega-y_\iota)^2}}\sqrt{(q-u)^2+(\omega-v)^2}\,dzd\omega+\sum_{\gamma=1}^{m}\sqrt{(z_\gamma-u)^2+(y_\gamma-v)^2}\,\hat{H}_\gamma\,. \quad (2.8)$$

It is possible to find the minimum point *(u, v)* by a gradient method. Gradient of function (2.8) is given by expression:

$$\nabla g(u,v)=\begin{pmatrix}\frac{\partial}{\partial u}g(u,v)\\ \frac{\partial}{\partial v}g(u,v)\end{pmatrix}=$$

$$=\begin{pmatrix}-\sum_{\iota=1}^{w}h_i\int_{y_\iota-r_\iota}^{y_\iota+r_\iota}\int_{z_\iota-\sqrt{r_\iota^2-(\omega-y_\iota)^2}}^{z_\iota+\sqrt{r_\iota^2-(\omega-y_\iota)^2}}\left[(q-u)^2+(\omega-v)^2\right]^{-\frac{1}{2}}(q-u)dqd\omega-\sum_{\gamma=1}^{m}\left[z_\gamma-u)^2+(y_\gamma-v)^2\right]^{-\frac{1}{2}}(z_\gamma-u)\hat{H}_j\\ -\sum_{\iota=1}^{w}h_i\int_{y_\iota-r_\iota}^{y_\iota+r_\iota}\int_{z_\iota-\sqrt{r_\iota^2-(\omega-y_\iota)^2}}^{z_\iota+\sqrt{r_\iota^2-(\omega-y_\iota)^2}}\left[(q-u)^2+(\omega-v)^2\right]^{-\frac{1}{2}}(\omega-v)dqd\omega-\sum_{\gamma=1}^{m}\left[(z_\gamma-u)^2+(y_\gamma-v)^2\right]^{-\frac{1}{2}}(y_\gamma-v)\hat{H}_j\end{pmatrix}.$$

Note, that we must apply gradient method carefully, because our objective function has ruptures on borders of districts. Therefore we have entered some changes into standard algorithm of a gradient method. As we cannot trust a gradient method completely, we have also checked it up by means of minimization function. Experiments have shown efficiency of a method at a small step.

Special software in Mathcad 14 has been developed. The complex of the programs allows calculating the population's density of Latvia for the given coordinate (*z, y)* by the formula (2.7). For example, for some coordinates (*z, y*) these densities are presented in Table 2.5.

Table 2.5

Population densities for coordinates *(z,y)*

Coordinate, *(z,y)*	190, 150	360, 80	320, 160	40, 100	100, 180
Density, *f(z,y)*	52.216	33.16	12.82	29.04	16.99

Thus, we see that the weakest coordinate on density is (320, 160). It is situated on the borders of Madona and Gulbene districts.

Table 2.6

Value of the objective function for different coordinates

Coordinate, *(u, v)*	190, 150	360, 80	320, 160	40, 100	100, 180
Objective function, *g(u, v)*	$1.856 * 10^8$	$3.946 * 10^8$	$3.184 * 10^8$	$4.284 * 10^8$	$3.228 * 10^8$

As we see the minimum value of objective function is found at the point (190, 150).

This complex of programs allows define the weighted average coordinate of Latvia by using a gradient method too. Calculations have shown that the weight average coordinate of Latvia is (189.94, 149.80). These coordinates correspond to city of Riga.

CONCLUSION

A problem of working out the analytical dependences, allowing for the specified coordinate of territory of the Latvia to calculate corresponding population's density is considered. The elaborated algorithm of the problem solution is based on the gradient method. A weighted average centre of Latvia is found. The considered numerical examples show its efficiency.

2.2. Mathematical model for problem of spatial location of service objects

The formal description of the problem is the following. Let us consider a real space *X* for that concrete *point* will be marked by *x*, for plane it is two-dimensional vector (it is available to consider another dimension too). *A distance* $l(x, x^*)$ is determined for points *x* and *x**, that satisfies usual conditional of distance axioms: $l(x, x) = 0$, $l(x, x^*) \geq 0$, $l(x, x^*) \leq l(x, x') + l(x', x^*)$.

Some *objects* are located in the space (for example men, animals, sport facilities). Let us name as ***x*****–*object***, the object that is at the point *x*. A density of object location is described by known density function $f(x) \geq 0$, so

$$\int_{x \in X} f(x)x = 1.$$

Some *service objects* must be located in the space, their number is k. It is necessary to determine those coordinates $x^{(1)}, x^{(2)}, ..., x^{(k)}$. If a x–object is serviced by i–th object then corresponding loss is equal to $g_x\left(x^{(i)}\right)$, for example $g_x\left(x^{(i)}\right) = g(x - x^{(i)})$. Let us call $g_x(\circ)$ as *loss function* and suppose that it is symmetrical according to zero: ($g_x(x^{(i)}) = g_x(-x^{(i)})$) and convex (down).

Probability that the x–object service that belongs to the *i-th* object is

$$\delta_i(x) = \frac{l\left(x, x^{(i)}\right)}{\sum_j l\left(x, x^{(j)}\right)}.$$

Now a problem can be formulated as follows: to find coordinates $x^{(1)}, x^{(2)}, ..., x^{(k)}$ of service object location that minimizes the total loss

$$D\left(x^{(1)}, x^{(2)}, ..., x^{(k)}\right) = \int \frac{1}{\sum_{i=1}^{k} l\left(\left(x, x^{(i)}\right)\right)^{-1}} \sum_{i=1}^{k} \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) f(x) dx. \qquad (2.9)$$

We union unknown variables of interest as vector $x = x^{(1)}, x^{(2)}, ..., x^{(k)}$.

Further let us use the following distance function and loss function:

$$l(x, t) = \sqrt{(x_1 - t_1)^2 + (x_2 - t_2)^2}, \qquad (2.10)$$

$$g(x, t) = |x_1 - t_1| + |x_2 - t_2|. \qquad (2.11)$$

2.3. Methods of optimization

2.3.1. Gradient optimization

Let the coordinates of an object be $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T \in R^2 = (-\infty, \infty) \times (-\infty, \infty)$, coordinates of the j-st object be $x^{(j)} = \begin{pmatrix} x_1^{(j)} & x_2^{(j)} \end{pmatrix}^T$. We will use a gradient method for the minimization of criteria (2.9). For that aim let us calculate a corresponding gradient. For a partial gradient with respect to the *j*-th object we have the following expression:

$$\frac{\partial}{\partial x^{(j)}} D\left(x^{(1)}, x^{(2)}, \dots, x^{(k)}\right) = \left(\frac{\partial}{\partial x_1^{(j)}} D\left(x^{(1)}, x^{(2)}, \dots, x^{(k)}\right) \quad \frac{\partial}{\partial x_2^{(j)}} D\left(x^{(1)}, x^{(2)}, \dots, x^{(k)}\right) \right)^T ,$$

where

$$\frac{\partial}{\partial x_q^{(j)}} D\left(x^{(1)}, x^{(2)}, \dots, x^{(k)}\right) = \frac{\partial}{\partial x_q^{(j)}} \int \frac{1}{\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1}} \sum_{i=1}^{k} \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) f(x) dx =$$

$$= \int \left(\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} \right)^{-1} \left(\left(g_x(x^{(j)}) \left(-\left(l\left(x, x^{(j)}\right)\right)^{-2} \right) \frac{\partial}{\partial x_q^{(j)}} l\left(x, x^{(j)}\right) + \left(l\left(x, x^{(i)}\right)\right)^{-1} \frac{\partial}{\partial x_q^{(j)}} g_x(x^{(j)}) \right) \right) f(x) dx - \quad (2.12)$$

$$- \int \left(\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} \right)^{-2} \left(\sum_i \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) \left(-\left(l\left(x, x^{(j)}\right)\right)^{-2} \right) \frac{\partial}{\partial x_q^{(j)}} l\left(x, x^{(j)}\right) \right) f(x) dx$$

Now we can present the gradient of (2.9) as ($2 \times k$)-matrix of the partial derivatives

$$\nabla D(x^{(1)}, x^{(2)}, \dots, x^{(k)}) = \begin{pmatrix} \frac{\partial}{\partial x_1^{(1)}} D(x^{(1)}, x^{(2)}, \dots, x^{(k)}) & \dots & \frac{\partial}{\partial x_1^{(k)}} D(x^{(1)}, x^{(2)}, \dots, x^{(k)}) \\ \frac{\partial}{\partial x_2^{(1)}} D(x^{(1)}, x^{(2)}, \dots, x^{(k)}) & \dots & \frac{\partial}{\partial x_2^{(k)}} D(x^{(1)}, x^{(2)}, \dots, x^{(k)}) \end{pmatrix}. \quad (2.13)$$

Then we have the following derivatives:

$$\frac{\partial}{\partial t_q} l(x,t) = -\frac{1}{\sqrt{(x_1 - t_1)^2 + (x_2 - t_2)^2}} (x_i - t_i), \quad (2.14)$$

$$\frac{\partial}{\partial t_q} g(x,t) = \frac{\partial}{\partial t_q} g\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right) = \begin{cases} 1 \ \ if \ \ x_i < t_i, \\ -1 \ \ otherwise. \end{cases} \quad (2.15)$$

For the optimization we use the two-stage procedure. At the first stage the component-wise (coordinate-wise) optimization is used as follows. During the *j*-th iteration (*j* = 1, 2, …, *k*) function (2.9) is minimized with respect to both coordinates of the *j*-st object $x^{(j)} = \left(x_1^{(j)} \quad x_2^{(j)}\right)$, at the same time other coordinates do not change. According to the gradient method we move along the gradient with respect to $\left(x_1^{(j)} \quad x_2^{(j)}\right)$, recalculating the one continually. At the second stage we work with the full gradient (2.13).

2.3.2. Quasi-Newton methods. The BFGS method

Quasi-Newton methods, like steepest descent, require only the gradient of the objective function $D(x) = D\left(x^{(1)}, x^{(2)}, ..., x^{(k)}\right)$ to be supplied at each iterate. By measuring the changes in gradients, they construct a model of the objective function that is good enough to produce superlinear convergence.

The most popular quasi-Newton algorithm is the BFGS method, named for its discoverers Broyden, Fletcher, Goldfarb, and Shanno.

The search direction p_k at stage k is given by

$$p_k = -B_k^{-1} \nabla D_k,$$

where B_k is an $n \times n$ symmetric positive definite matrix that will be revised or *updated* at every iteration;

Let initial value x_0 is fixed. ∇D_k is $(2 \times k)$-matrix of the partial derivatives of the function (2.9);

The new iterate determines

$$x_{k+1} = x_k + \alpha_k p_k,$$

where the step length α_k is chosen to satisfy the Wolfe conditions.

The Wolfe conditions is a popular inexact line search condition stipulates that α_k should first of all give *sufficient decrease* in the objective function D, as measured by the following inequality:

$$D(x_k + \alpha p_k) \le D(x_k) + c\alpha_k \nabla D_k^T p_k$$

for some constant $c \in (0, 1)$ (typical values of c are 0.9 when the search direction p_k is chosen by a Newton or quasi-Newton method). In other words, the reduction in D should be proportional both the step length α_k and the directional derivative $\nabla D_k^{\,T} p_k$.

To realize BFGS method step length α_k is computed from a line search procedure to satisfy the Wolfe conditions.

Algorithm 1 (Backtracking line search)

Choose $\bar{\alpha} > 0$, $\rho \in (0, 1)$, $c \in (0, 1)$; Set $\alpha \leftarrow \bar{\alpha}$
repeat until $D(x_k + \alpha p_k) \le D(x_k) + c\alpha_k \nabla D_k^T p_k$
Set $\alpha \leftarrow \rho\alpha$;
end (repeat)
Terminate with $\alpha_k = \alpha$

In this procedure, the initial step length $\bar{\alpha}$ is chosen to be 1 in Newton and quasi-Newton methods. In practice, the contraction factor ρ is often allowed to vary at each iteration of the line search.

We can derive a version of the BFGS algorithm that works with the Hessian approximation B_k.

Algorithm 2 (BFGS Method)

Given starting point x_0, convergence tolerance $\varepsilon > 0$,
inverse Hessian approximation B_0;
$k \leftarrow 0$;
while $\|\nabla D_k\| > \varepsilon$;
Compute ∇D_k and search direction

$$p_k = -B_k^{-1} \nabla D_k$$

Set $x_{k+1} = x_k + \alpha_k p_k$ where α_k is computed from a line search procedure to satisfy the Wolfe conditions;
Define $s_k = x_{k+1} - x_k$ and $\zeta_k = \nabla D_{k+1} - \nabla D_k$;

Compute $$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\zeta_k \zeta_k^T}{\zeta_k^T s_k}; \tag{2.16}$$

Set $k \leftarrow k + 1$;
end (while)

The algorithm is robust, and its rate of convergence is superlinear, which is fast enough for most practical purposes. Even though Newton's method converges more rapidly (that is, quadratically), its cost per iteration usually is higher.

Tables 2.7 and 2.8 contain the results of sequential cycles of the optimization procedure.

Table 2.7

Results of gradient optimization

	Iteration number				
	1	2	3	4	5
$x_1^{(1)}$	40	41.65	43.24	44.78	46.33
$x_2^{(1)}$	80	82.57	85.08	87.72	90.38
$x_1^{(2)}$	100	103.22	106.48	109.79	113.17
$x_2^{(2)}$	130	131.54	133.05	134.39	135.73
$x_1^{(3)}$	60	61.63	63.18	64.65	66.16
$x_2^{(3)}$	60	62.17	64.34	66.47	68.70
$x_1^{(4)}$	190	196.53	203.16	209.98	216.74
$x_2^{(4)}$	100	105.26	110.47	115.45	120.39
D	$2.24 * 10^8$	$2.19 * 10^8$	$2.11 * 10^8$	$2.04 * 10^8$	$1.98 * 10^8$

Table 2.8

Results of the BFGS method

	Iteration number				
	1	2	3	4	5
$x_1^{(1)}$	40	40.16	32.24	70.98	68.61
$x_2^{(1)}$	80	80.25	96.67	77.74	93.93
$x_1^{(2)}$	100	100.32	131.69	157.11	165.91
$x_2^{(2)}$	130	130.15	184.13	103.24	114.68
$x_1^{(3)}$	60	60.16	57.16	92.99	93.76
$x_2^{(3)}$	60	60.21	78.57	71.16	83.64
$x_1^{(4)}$	190	190.65	225.14	291.56	289.96
$x_2^{(4)}$	100	100.52	196.24	35.93	60.91
α	1	1	1	0.5	0.5
D	$2.24 * 10^8$	$2.23 * 10^8$	$2.02 * 10^8$	$2.07 * 10^8$	$1.88 * 10^8$

From the tables we can see how the gradient and BFGS methods improves the criteria value continually. As we can see, both methods are of equal value; therefore we can use gradient method as a simpler.

CONCLUSION

A problem of service object location in spatial space is considered. The elaborated algorithms of the problem solution are based on the gradient and quasi-Newton methods. The considered numerical examples show its efficiency.

2.3.3. Genetic algorithm

The genetic algorithm is a method for solving both constrained and unconstrained optimization problems that is based on natural selection, the process that drives biological evolution.

Every current considered vector $\tilde{x} = \left(x^{(1)}, x^{(2)},..., x^{(k)}\right)$ is called an *acceptable chromosome*. The acceptability means that all elements of satisfies to condition (2.24). Large set of chromosomes is called a *generation*, it's denoted by *n*. In other words, generation is a set $P = \{\tilde{x}_1, \tilde{x}_2,..., \tilde{x}_n\}$ [6].

The genetic algorithm uses three main types of rules at each step to create the next generation from the current population:

- *Selection rules* select the chromosomes from current generation, called parents, to form the next generation.
- *Crossover rules* combine two parents from current generation to form new chromosome (children) for the next generation.
- *Mutation rules* improve individual chromosome by random changes of chromosome element [25].

The traditional genetic algorithm uses single point *crossover* where the two mating chromosomes $\tilde{x}_1 = (x_1^{(1)}, x_1^{(2)},..., x_1^{(k)})$ and $\tilde{x}_2 = (x_2^{(1)}, x_2^{(2)},..., x_2^{(k)})$ are cut once at corresponding points and the sections after the cuts are exchanged.

Here, the cross-site or crossover point *R* is selected randomly along the length of the mated strings within the interval [1, 2, …, *k*]. In the point *R*, chromosomes divide into two parts: $\tilde{x}_1 = (x_1^{(1)}, x_1^{(2)},..., x^{(R)}, x^{(R+1)},..., x_1^{(k)})$, $\tilde{x}_2 = (x_2^{(1)}, x_2^{(2)},..., x_2^{(R)}, x_2^{(R+1)},..., x_2^{(k)})$. Finally, the position values are swapped between the two strings following the cross site. Thus new chromosomes are obtained:

$$\tilde{\tilde{x}}_1 = (x_1^{(1)}, x_1^{(2)},..., x_1^{(R)}, x_1^{(R+1)},..., x_1^{(k)}),\ \tilde{\tilde{x}}_2 = (x_2^{(1)}, x_2^{(2)},..., x_2^{(R)}, x_2^{(R+1)},..., x_2^{(k)}). \qquad (2.17)$$

For example, if R=5, k=11 and

$$\tilde{x}_1 = (10110011100), \quad \tilde{x}_2 = (01100110011),$$

then

$$\tilde{x}_1 = (10110110011), \quad \tilde{x}_2 = (01100011100).$$

Mutation is a genetic operation for algorithm chromosome $\tilde{x} = (x^{(1)}, x^{(2)},..., x^{(k)})$ aiming to maintain genetic diversity. More precisely, mutation changes $x^{(R)}$ value 0 or 1 of the *R-th* component by the opposite value $\bar{x}^{(R)}$:

$$\tilde{\tilde{x}} = (x^{(1)}, x^{(2)},..., x^{(R-1)}, \bar{x}^{(R)}, x^{(R+1)},..., x^{k}). \qquad (2.18)$$

R value is selected randomly. For example, if R = 2 and $\tilde{x} = (10110110011)$ is the last obtained chromosome, after the mutation

$$\tilde{\tilde{x}} = (11110110011).$$

Selection operation is meant for the selection of population's chromosomes for reproduction. The fitter is the chromosome, the more times it is likely to be selected for reproduction. Chromosome $\tilde{x}_j = (x_j^{(1)}, x_j^{(2)},..., x_j^{(k)})$ from the old generation is selected for the new generation due its objective function value *D(x)* see formula (2.9). In case of maximization, probability p_j, that chromosome $\tilde{x}_j$ is chosen, is calculated as

$$p_j = D(\tilde{x}_j) / \sum_{v=1}^{n} D(\tilde{x}_v). \qquad (2.19)$$

In case of minimization problem:

$$p_j = \frac{1}{D(\tilde{x}_j)} \left(\sum_{v=1}^{n} \frac{1}{D(\tilde{x}_v)} \right)^{-1}. \qquad (2.20)$$

To create a new generation, it is necessary to make n independent random attempts. In each attempt, one chromosome will be chosen for the new generation. The uniformly distributed random variable R from the interval (0, 1) is generated. The chromosome under number j will be chosen if the following inequality takes place:

$$F_{j-1} < R \le F_j, \quad j = 1,2,...,n, \tag{2.21}$$

where $F_0 = 0,\ F_n = 1$ and

$$F_j = \sum_{v=1}^{j} p_v, \quad j = 1,\ 2,...,n-1.$$

The same chromosome can be chosen many times in the new generation.

After the generation is formed, crossover and mutation take place. Furthermore, crossover probability p_c and mutation probability p_m are provided. The mutation probability determines how often the elements of a chromosome will be mutated. Crossover probability is a parameter determining how often crossover will be performed. Crossover probability and mutation probability are the main factors affecting balanced search ability (global search ability and local search ability).

Crossover starts with the selection of chromosomes for this operation. First, independent attempts n are made. In current attempt, the uniformly distributed random variable R from interval (0, 1) is generated. If

$$R \le p_c, \tag{2.22}$$

then chromosome $\tilde{x}_j$ is chosen for crossover. Let *sk* be a number of chosen chromosomes.

Then the chromosomes are grouped in pairs and the crossover is performed (2.17). Then the mutation is performed. Independent attempts are repeated by n times. In current attempt, the uniformly distributed random variable R from interval (0, 1) is generated. If

$$R \le p_m, \tag{2.23}$$

the chromosome is chosen for mutation (2.18). Let *sm* be a number of chosen chromosomes.

Below we describe corresponding algorithm.

Input data:

1) objective function (2.9) for optimization;

2) considered set T from (2.24);

3) algorithm parameters: n – size of population; h – number of investigated populations; p_c and p_m – crossover (we assume p_c =0.95) and mutation (we assume p_m=0.01) probabilities.

Output data: the best solution to the problem found by the genetic algorithm.

Algorithm 3

1) Start with randomly generated population, so-called *old population* (*OP*) as *k* x *n* matrix. Assume that *t=1*.
2) Continue if $t \leq h$.
   a) *Selection*. With the help of *selection*, by using old population *OP*, create new population *NP* as *k* x *n* matrix.
   b) *Crossover*.
      i) Create (by using formula (2.22)) a vector $V=(V_1, V_2, \ldots, V_{sk})$, which contains the numbers of *sk* chromosomes from new population *NP*.
      ii) Continue if $sk \geq 2$.
      Perform crossover for chromosomes $\tilde{x}_{sk}$ and $\tilde{x}_{sk-1}$ and create new chromosomes $\tilde{\tilde{x}}_{sk}$ and $\tilde{\tilde{x}}_{sk-1}$. If $\tilde{\tilde{x}}_{sk}$ satisfies the conditions (2.24) and $D(\tilde{\tilde{x}}_{sk})$ value is better than $D(\tilde{x}_{sk})$, the chromosome $\tilde{x}_{sk}$ in generation *NP* should be substituted by $\tilde{\tilde{x}}_{sk}$. Repeat this procedure with $\tilde{\tilde{x}}_{sk-1}$ and $\tilde{x}_{sk-1}$. Set *sk*=*sk*-2.
   c) Mutation
      i) Create (by using formula (2.23)) a vector $W=(W_1, W_2, \ldots, W_{sm})$, which contains the numbers of *sm* chromosomes from new population *NP*.
      ii) Perform mutation for each chromosome $\tilde{x}_j$, where $j \in W$, and obtain chromosome $\tilde{\tilde{x}}_j$. If $\tilde{\tilde{x}}_j$ satisfies the conditions (2.24) and $D(\tilde{\tilde{x}}_j)$ value is better than $D(\tilde{x}_j)$, then the chromosome $\tilde{x}_j$ in generation *NP* should be substituted by $\tilde{\tilde{x}}_j$.
      iii) Set *OP*=*NP*, *t*=*t*+1 and go to step 2.
3) Finally we must select chromosome x^* from the last population *NP* with the best objective function value and accept it as a solution to the problem [6].

In our case we use function *D(x)* from (2.9) as objective function. A problem is formulated as follows: to find coordinates $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ of object location that minimizes the total loss

$$D\left(x^{(1)}, x^{(2)}, \ldots, x^{(k)}\right) = \int \frac{1}{\sum_{i=1}^{k} l\left(\left(x, x^{(i)}\right)\right)^{-1}} \sum_{i=1}^{k} \left(l\left(x, x^{(i)}\right)\right)^{-1} g_x(x^{(i)}) f(x) dx$$

on condition that

$$x^{(i)} \in T, \tag{2.24}$$

where T means a considered territory.

Further we union unknown variables of interest as vector $\tilde{x} = \left(x^{(1)}, x^{(2)}, ..., x^{(k)}\right)$. In our case the considered vector is the vector containing all coordinates (set of coordinates). Each coordinate is 2 numbers. Chromosome $\tilde{x} = (x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, x_2^{(2)}, ..., x_1^{(k)}, x_2^{(k)})$, where $(x_1^{(1)}, x_2^{(1)})$ coordinates of the 1-st service object, $(x_1^{(2)}, x_2^{(2)})$ coordinates of the 2-nd service object, $(x_1^{(k)}, x_2^{(k)})$ coordinates of the k-th service object.

A matrix A (see Figure 2.2) describes the territory of Latvia T. Each element of a matrix corresponds to an elementary square of Latvia territory which is perceived as a point (pair of coordinates $(x_{1,i}, x_{2,j})$).

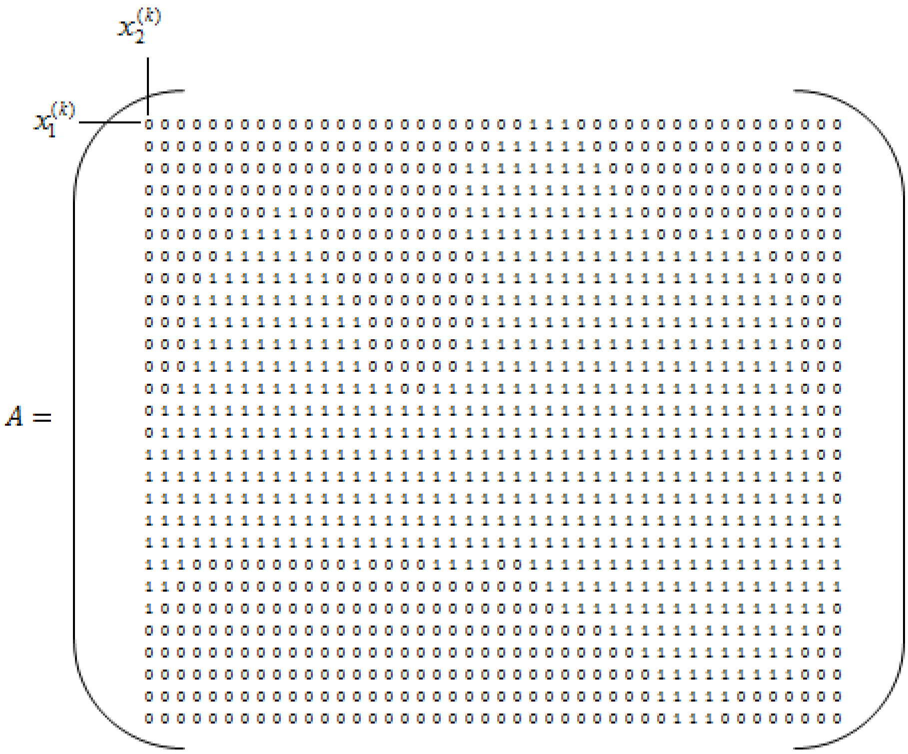

Figure 2.2 Matrix A

The matrix A contains the following information:

$$A = \begin{cases} 1, \text{ if coordinates } (x_{1,i}^{(k)}, x_{2,j}^{(k)}) \text{ belong to the territory of Latvia,} \\ 0, \textit{ otherwise.} \end{cases}$$

Therefore acceptable condition (2.24) for point $x^{(v)} = (x_1^{(v)}, x_2^{(v)}) = (x_{1,i}, x_{2,j})$ is satisfied if $A_{i,j} = 1$.

Special programs in MATLAB R2009b have been developed. The complex of the programs allows optimize objective function (2.9) using genetic algorithm (Algorithm 3).

Ten experiments with different data have been made.

Table 2.9

Overall results of the experiments

	$x_1^{(1)}$	$x_2^{(1)}$	$x_1^{(2)}$	$x_2^{(2)}$	$x_1^{(3)}$	$x_2^{(3)}$	$x_1^{(4)}$	$x_2^{(4)}$	D
Experiment Nr.1	356.07	88.23	223.32	240.16	121.71	149.57	120.21	169.95	$1.355*10^8$
Experiment Nr.2	281.82	193.94	316.75	144.44	98.64	168.61	108.42	140.61	$1.42*10^8$
Experiment Nr.3	259.28	82.88	338.03	101.86	235.19	151.56	100.08	82.55	$1.581*10^8$
Experiment Nr.4	308.51	69.19	262.68	254.46	167.39	121.61	60.328	135.21	$1.485*10^8$
Experiment Nr.5	312.77	200.88	134.49	182.98	191.22	157.82	301.90	75.87	$2.018*10^8$
Experiment Nr.6	292.74	239.88	287.96	89.80	76.70	96.68	68.05	103.77	$1.702*10^8$
Experiment Nr.7	252.45	204.64	258.57	116.63	117.52	147.01	166.07	123.40	$1.438*10^8$
Experiment Nr.8	284.12	227.05	190.30	153.10	161.50	134.91	163.03	91.54	$1.486*10^8$
Experiment Nr.9	344.01	205.073	266.68	74.60	94.96	135.02	175.912	143.33	$1.324*10^8$
Experiment Nr.10	232.52	98.51	283.07	180.0	102.57	113.39	215.02	138.34	$1.468*10^8$

On the Table 2.9 points are the coordinates of the optimal decision and D is the meaning of the objective function. From the table we can see how the genetic algorithm improves the criteria value continually.

The genetic algorithm yields the best results, than gradient method. The reason in that is the following: gradient method ends work in a local minimum. Our problem is multiextreme one, therefore last case constantly repeats. The genetic algorithm not so strongly suffers from multiextremeness. The best results turn out by a combination of both methods. On the Table 2.10 we see results of the gradient method, started from points, received by genetic algorithm.

Table 2.10

Combination of the genetic algorithm and gradient method

Optimization methods	$x_1^{(1)}$	$x_2^{(1)}$	$x_1^{(2)}$	$x_2^{(2)}$	$x_1^{(3)}$	$x_2^{(3)}$	$x_1^{(4)}$	$x_2^{(4)}$	D
Genetic Algorithm	232.52	98.51	283.07	180.0	102.57	113.39	215.02	138.34	$1.468*10^8$
Gradient method	244.30	116.62	292.96	174.69	103.82	128.33	224.91	142.14	$1.39*10^8$

From the table we can see that value of the objective function improves. At the further start of the gradient optimization value of the function didn't change. All received coordinates lie on territory of Latvia.

CONCLUSION

A problem of service object location in spatial space is considered. The elaborated algorithm of the problem solution is based on the combination of the genetic algorithm and gradient method. The considered numerical examples show its efficiency.

2.4. Application of the considered methods

There is track and field arenas shortage in the Latvia (see Figure 2.3). Today this is one of the actual sports facilities shortage problems.

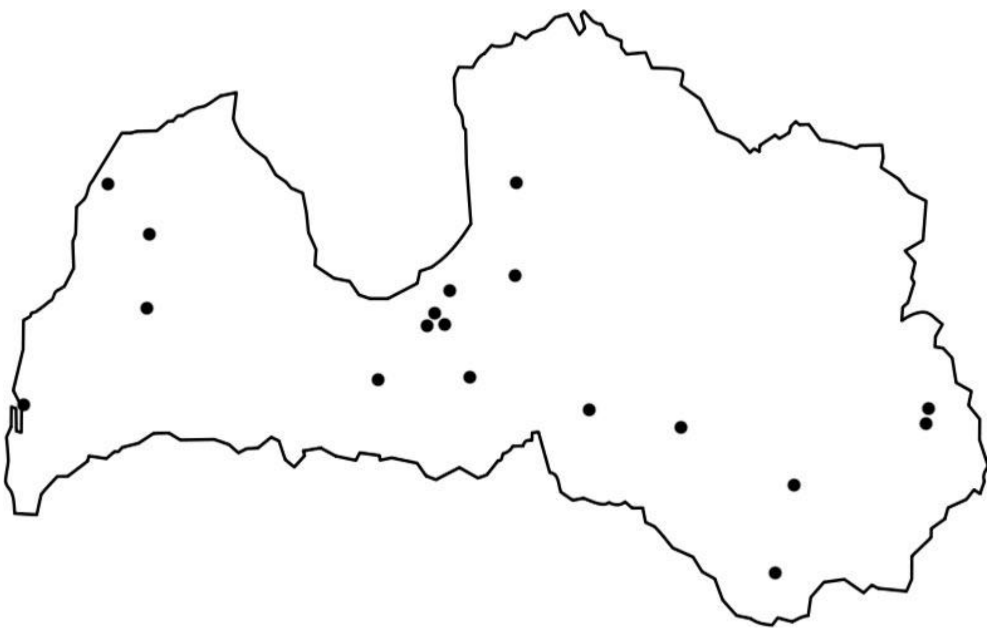

Figure 2.3 Location of existing track and field arenas

List and coordinates of the existing track and field arenas is given in the Table 2.11. Data is taken form the Ministry of Education and Science of the Republic of Latvia data base “*Sports facilities register*” (http://sportabazes.izm.gov.lv/sbdb/), which one was created in collaboration with the author of the doctoral thesis.

Table 2.11

Location and coordinates of the existing track and field arenas

Name of the sports facility	Location	Coordinate z	Coordinate y
Baldones sporta komplekss	Baldones novads	211	115,823
Biedrības “Latvijas Olimpiskā komiteja” olimpiskais sporta centrs	Rīga, Ziemeļu rajons	194,71	140,968
Daugavpils SP Bērnu jaunatnes sporta skola	Daugavpils	346,951	23,536
Jēkabpils 3.vidusskola	Jēkabpils novads	303,432	91,219
Jelgavas sporta halle	Jelgava	170,102	104,807
„Kuldīgas sporta aģentūras” vieglatlētikas manēža	Kuldīgas novads	63,441	142,434
Latvijas sporta pedagoģijas akadēmija	Rīga, Vidzemes priekšpilsēta	199,735	142,838
Liepājas sporta manēža	Liepāja	3,54	97,194
Ludzas novada Sporta skola	Ludzas novads	416,199	99,553
Murjāņu sporta ģimnāzija	Sējas novads	228,147	160,663
Preiļu 2.vidusskola	Preiļu novads	355,835	68,565
Rīgas pašvaldības sporta iestāde "Rīgas Nacionālā sporta manēža"	Rīga, Latgales priekšpilsēta	196,239	137,278
SIA "Olimpiskais centrs "Limbaži" "sporta komplekss	Limbažu novads	229,747	201,126
SIA "Olimpiskais centrs "Ventspils"" sporta komplekss	Ventspils	41,152	190,074
Sporta biedrība "Vārpa"	Aizkraukles novads	263,39	100,547
Sporta komplekss "Vārpa"	Ludzas novads	416,449	99,02
Ugāles vidusskola	Ventspils novads	67,471	177,187
VSIA "Kultūras un sporta centrs "Daugavas stadions""	Rīga, Latgales priekšpilsēta	196,74	139,302

According to this information we set our problem as following. We have request from the Sports Department to locate 4 new track and field arenas on the territory of Latvia. Final solution must depend from the distribution of population on the territory of Latvia. Corresponding data and analytical expressions were given in the Chapter 2.1.

For optimization genetic algorithm (Algorithm 3) is used. The following results of experiments are presented below. 10 experiments have been made. Each experiment corresponds to the new initial data. Received results are shown in the Table 2.12. Pair $(x_1^{(k)}, x_2^{(k)})$ determines the coordinates of the location of the *k-th* object. D is the value of the objective function.

Table 2.12

Overall results of the experiments

	$x_1^{(1)}$	$x_2^{(1)}$	$x_1^{(2)}$	$x_2^{(2)}$	$x_1^{(3)}$	$x_2^{(3)}$	$x_1^{(4)}$	$x_2^{(4)}$	D
Experiment Nr.1	299.64	223.65	336.06	99.49	204.53	147.73	112.34	121.50	$1.365*10^8$
Experiment Nr.2	326.42	62.61	259.02	198.14	117.28	123.25	175.80	136.60	$1.351*10^8$
Experiment Nr.3	254.50	224.45	361.68	64.48	136.34	154.27	112.18	128.87	$1.341*10^8$
Experiment Nr.4	345.01	121.24	253.81	225.09	95.07	144.89	164.73	140.55	$1.333*10^8$
Experiment Nr.5	344.90	64.92	244.91	225.32	104.12	134.88	135.13	164.83	$1.334*10^8$
Experiment Nr.6	354.12	114.43	275.27	225.09	94.79	139.85	170.20	135.64	$1.3272*10^8$
Experiment Nr.7	262.18	224.88	337.47	92.13	177.30	134.99	95.18	136.21	$1.329*10^8$
Experiment Nr.8	256.99	87.04	345.26	212.43	156.96	144.75	95.98	142.67	$1.332*10^8$
Experiment Nr.9	361.14	65.17	275.37	222.70	105.03	135.03	168.53	134.89	$1.332*10^8$
Experiment Nr.10	343.62	114.31	274.40	225.99	95.40	135.23	176.63	134.97	$1.3274*10^8$

From the table we can see how the genetic algorithm improves the criteria value continually. Experiment Nr. 6 gives the best results (see Figure 2.4). Here the new track and field arenas indicated with white points.

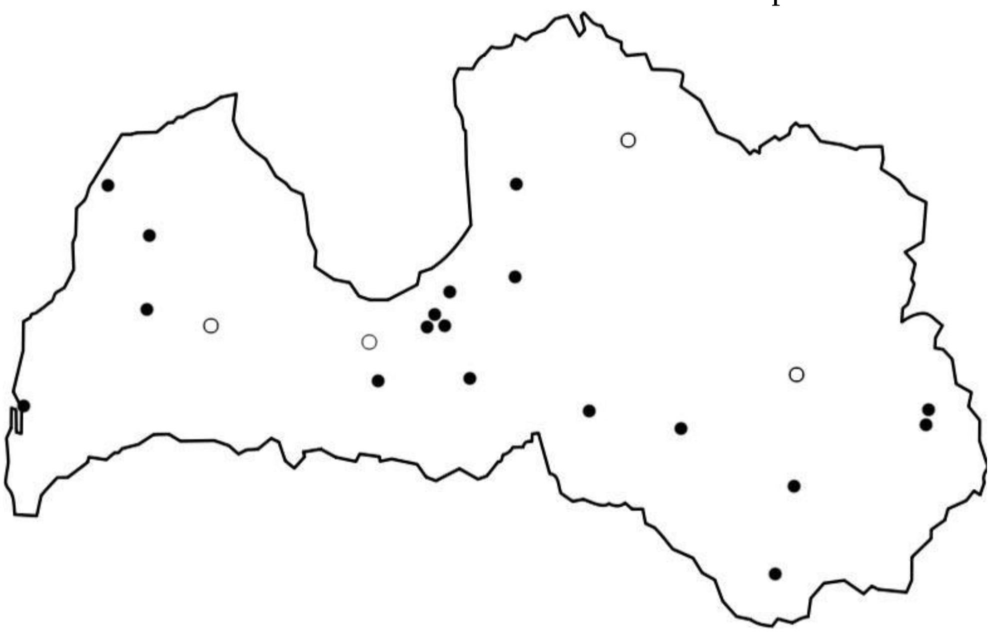

Figure 2.4 Location of the new track and field arenas

A problem of track and field arenas on the territory of Latvia is considered. The elaborated algorithm of the problem solution is based on the genetic algorithm. Special program in MATLAB R2009b has been developed. The considered numerical examples show its efficiency. Our experience shows that the program works well, and the received results are approved by the requester.

3. CONCLUSIONS

I. SCIENTIFIC NOVELTY AND MAIN RESULTS

1. Doctoral thesis is devoted to development of the mathematical models, methods, algorithms and computer programs for the problem of optimal location of service objects within the existing transport infrastructure, investigation of the elaborated methods efficiency and obtained results application to practical problem solution. As nowadays an optimal location of service objects is practically very important problem, the presented work is actual one.
2. All the tasks defined in the beginning of the doctoral thesis were completed.
3. Analysis of the statistical data about population distribution on the territory of Latvia has been performed. As result an analytical expression for population density has been obtained. On this basis the following problem is solved: a weighted average centre of Latvia is found, i.e. coordinate, for which the total distance from all points of Latvia (subject to density) is minimal one. Necessary statistical information has been received from Central Statistical Bureau of Latvia.
4. The original mathematical model for problem of optimal spatial location of service objects has been developed. Modern optimization methods and algorithms such a line search methods (step length, the Wolfe conditions, backtracking line search), gradient optimization, quasi-Newton methods (the BFGS method), and genetic algorithm are used. Special computer programs in Mathcad and MATLAB environment for these methods has been elaborated.
5. The developed model and software are suitable for any interpretation of “transport expenses”. The only essential is that these expenses depend (possibly nonlinearly) on the distance between the dislocation of a customer and an object. There can be the following distances (depending on the conditions of the task being considered): geographical distance, distance in kilometres within the existing road network, time necessary to overcome the required distance, fare.
6. The experimental research of efficiency of the developed methods and programs has been performed. Advantage of genetic algorithm in comparison with the algorithms based on a method of a gradient, for multiextreme problems, is shown. It is established that the best results are given by a combination of genetic algorithm and a gradient method. It was stated, that the best results are obtained, when the initial points for optimization are chosen by experts.

II. APPROBATION

1. Developed models and methods of the description of a population distribution on the territory of Latvia were used in the scientific project "Creation of mathematical models, algorithms and computer programs for Latvia's transport system's analysis, development prognosis and optimization", which was a component of the Scientific Project "Zinātniskās darbības attīstība augstskolās" and lasted from June 1 till December 31, 2008.
2. On the basis of the obtained results a part of lectures and practical works on the study discipline "Computer Methods for Engineering Problem Solving" for the first year student of master's studies programme of the Riga Technical University Institute of Transport Vehicle Technologies is prepared.
3. The problem of the optimal location of the new track and field arenas on the territory of Latvia is solved in collaboration with the Sports Department of the Ministry of Education and Science of the Republic of Latvia. Statistical data are taken from the data base "Sports facilities register" maintained by the Ministry of Education and Science of the Republic of Latvia, which was created in collaboration with the author of the doctoral thesis.
4. The obtained models and algorithms can be used by private and government companies for the optimal location of the service objects, i.e. car technical maintenance objects, sports facilities, filling objects and others.
5. The main results of the present investigation are published in 8 articles and presented at 9 international scientific conferences in Latvia, France and Italy.

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