Strong-Stability-Preserving, k-Step, 4- to 10-Stage, Hermite–Birkhoff Time-Discretizations of Order 4

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Abstract – **Optimal *k*-step, 4- to 10-stage, explicit, strong-stability-preserving Hermite–Birkhoff (SSP HB) methods of order 4 with nonnegative coefficients are constructed by combining linear *k*-step methods with 4- to 10-stage Runge–Kutta (RK) methods of order 4. These new methods preserve the monotonicity property and prevent the growth of error; therefore, they are suitable for solving hyperbolic PDEs by the method of lines. Moreover, the series of new HB methods have larger effective SSP coefficients and larger maximum effective CFL numbers than Huang's hybrid methods of the same order and RK methods of the same stage number and the same order when applied to inviscid Burgers' equations.**

Keywords **– Strong-stability-preserving method; Hermite–Birkhoff method; SSP coefficient; time discretization; method of lines; comparison with other SSP methods.**

I. Introduction

In our research, we are concerned with the numerical solution of nonlinear time-dependent partial differential equations (PDEs) by the method of lines. Here the spatial discretization yields a set of time-dependent ODEs with initial condition of the form

$$\frac{dy}{dt} = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1}$$

where $y \in \mathbb{R}^N$ is the semi-discrete state and $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ represents the discretization of the spatial variables forming a system of semi-discrete equations.

The development of strong-stability-preserving (SSP) methods is motivated by the observation of the solutions of PDEs. Indeed, the solutions of many important PDEs satisfy the monotonicity property

$$\|y(t + \Delta t)\| \le \|y(t)\|, \tag{2}$$

where $\|\cdot\|$ is a norm, semi-norm, or, more generally, any convex functional. Therefore, when applying a spatial discretization to PDEs to obtain (1), it is important to keep the discrete monotonicity analogous to (2), i.e.,

$$\|y_{n+1}\| \le \|y_n\|, \tag{3}$$

where y_n is a numerical approximation to $y(t_0 + n\Delta t)$.

Here, we assume that the semi-discretization is designed so that the solution of the resulting ODE system (1) satisfies the monotonicity property analogous to (2) under the forward Euler method (FE), i.e.,

$$\|y_n + \Delta t f(t_n, y_n)\| \le \|y_n\|, \tag{4}$$

for all $0 < \Delta t \le \Delta t_{\rm FE}$ and all y_n where $\Delta t_{\rm FE}$ is a maximal step size for which (4) holds, and $\|\cdot\|$ is a given convex functional.

Under this assumption, we are interested in higher-order, explicit Hermite–Birkhoff (HB) methods, that is, a class of multistep and multistage numerical methods which preserve the monotonicity property

$$\|y_n\| \le \max_{1\le j\le k} \|y_{n-j}\|, \quad 0 \le \Delta t \le c\Delta t_{\rm FE}, \tag{5}$$

when coupled with the spatial semi-discretization. Here *k* is a positive integer representing the number of previous steps used to compute the numerical solution at the next step. The monotonicity property (5) is desirable since it mimics property (2) and prevents approximations from error growth.

A method, which gives the solution satisfying monotonicity property (5) for (1) which satisfies the FE condition (4) is called the SSP method. The number *c* is referred to the SSP coefficient, which depends only on the numerical method. In the literature, *c* has been denoted by the CFL coefficient. However, SSP coefficient is a more suitable term because the CFL condition is derived from the ratio between the time step and the spatial grid size, while the SSP coefficient describes the ratio between the SSP time step and the strongly stable FE time step. A great effort has been devoted to search for the numerical methods with the largest SSP coefficients (see [3], [4] and [6]).

In this paper, we concentrate on constructing new explicit, multistep, multistage, SSP general linear time-discretization methods of order 4 with nonnegative coefficients in order to solve system (1) by discretizing one-dimenional conservation laws:

$$y_t + g(y)_x = 0, \quad y(x, 0) = y_0(x), \tag{6}$$

by finite differences or finite elements (see, for example, [1], [7], [10], [17]).

There are reasons to call these methods SSP HB methods. Firstly, the construction of new methods involves HB interpolation polynomials, and they are combinations of linear *k*-step methods and *s*-stage Runge–Kutta (RK) methods of order 4, called HB*ks*. Secondly, by convexity and using the extension of Shu–Osher representations, we can decompose the method in terms of SSP FE method, so these methods will maintain the monotonicity property, while having high-order accuracy in time, perhaps under a modified restriction

$$\Delta t \le c(\mathrm{HB}ks)\Delta t_{\rm FE}. \tag{7}$$

The paper is organized as follows. Section II introduces 4- to 10-stage SSP HB methods. Order conditions are listed in Section III. In Section IV, we derive the Shu–Osher representation of *k*-step, 4- to 10-stage HB*ks* of order 4. The new methods are formulated as solutions of optimization problems in Section V. Section VI compares the effective SSP coefficients of our new SSP HB methods with other known SSP methods. In Section VII, numerical results are presented for several methods when applied to inviscid Burgers' equations. The new methods are listed in Appendix A in their Shu–Osher form.

II. SSP HB Methods with *k* Steps and *s*-Stages of Order 4 and Some Notation

Throughout this paper, the following notation will be used:

Notation 1:

- *k* denotes the number of steps of a given method,
- *s* denotes the number of stages of a given method,
- HB*ks* denotes *k*-step, *s*-stage SSP Hermite–Birkhoff methods of order 4,
- GL*ks* denotes *k*-step, *s*-stage general linear methods,
- HM*k* denotes *k*-step SSP hybrid methods of order 4,
- RK*s* denotes *s*-stage SSP Runge–Kutta methods of order 4.

All the methods considered in this paper are SSP and are of order 4, so the denominations "SSP" and "order 4" will generally be omitted in what follows.

Notation 2:

- The abscissa vector $\sigma = [c_1, c_2, c_3, \dots, c_s]^T$, $0 \le c_j \le 1$, defines the off-step points $t_n + c_j\Delta t$. For simplicity, set $c_1 = 0$ and $c_1^0 = 1$ by convention.
- At each off-step point, let $F_j := f(t_n + c_j\Delta t, Y_j)$ be the *j*th-stage derivative where Y_j is the *j*th-stage value and set $Y_1 = y_n$.

An HB*ks* method includes the following *s* formulae to perform integration from t_n to t_{n+1}.

An HB polynomial of degree $2k+i-3$ is used as predictor P_i to obtain the *i*th-stage value Y_i,

$$Y_i = \sum_{j=0}^{k-1} \alpha_{ij} y_{n-j} + \Delta t\left[\sum_{j=1}^{i-1} a_{ij}F_j + \sum_{j=1}^{k-1} \beta_{ij} f_{n-j}\right], \quad i = 2,3,\dots,s. \tag{8}$$

An HB polynomial of degree $2k+s-2$ is used as the integration formula to obtain y_{n+1} to order 4:

$$y_{n+1} = \sum_{j=0}^{k-1} \alpha_j y_{n-j} + \Delta t\left[\sum_{j=1}^{s} b_j F_j + \sum_{j=1}^{k-1} \beta_j f_{n-j}\right]. \tag{9}$$

III. Order Conditions of HB*ks*

For the construction of the order conditions for HB*ks*, we shall use the following expressions derived from the backsteps of the methods:

$$B_i(j) = \sum_{\ell=1}^{k-1} \alpha_{i\ell} \frac{(-\ell)^j}{j!} + \sum_{\ell=1}^{k-1} \beta_{i\ell} \frac{(-\ell)^{j-1}}{(j-1)!}, \quad j = 1,2,\dots,4, \quad i = 2,3,\dots,s. \tag{10}$$

Like with Runge–Kutta methods, we impose the following simplifying conditions on the abscissa vector $\sigma = [c_1, c_2, c_3, \dots, c_s]^T$

$$c_i = \sum_{j=1}^{i-1} a_{ij} + B_i(1), \quad i = 2,3,\dots,s. \tag{11}$$

Expanding the numerical solution produced by formulae (8)-(9) to agree with a Taylor expansion of the true solution, we obtain the following multistep- and RK-type order conditions that must satisfy with HB*ks*:

$$\sum_{j=0}^{k-1} \alpha_{ij} = 1, \quad i = 2,3,\dots,s, \tag{12}$$

$$\sum_{i=0}^{k-1} \alpha_i = 1, \tag{13}$$

$$\sum_{i=1}^{s} b_i c_i^m + m!B(m+1) = \frac{1}{m+1}, \quad m = 0,1,2,3, \tag{14}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}c_j + B_i(2)\right] + B(3) = \frac{1}{3!}, \tag{15}$$

$$\sum_{i=2}^{s} b_i \frac{c_i}{3}\left[\sum_{j=1}^{i-1} a_{ij}c_j + B_i(2)\right] + B(4) = \frac{1}{4!}, \tag{16}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}\frac{c_j^2}{2!} + B_i(3)\right] + B(4) = \frac{1}{4!}, \tag{17}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}\left[\sum_{k=1}^{j-1} a_{jk}c_k + B_j(2)\right] + B_i(3)\right] + B(4) = \frac{1}{4!}, \tag{18}$$

where the backstep parts, *B*(*j*), are defined by

$$B(j) = \sum_{i=1}^{k-1} \alpha_i \frac{(-i)^j}{j!} + \sum_{i=1}^{k-1} \beta_i \frac{(-i)^{j-1}}{(j-1)!}, \quad j = 1,2,3,4. \tag{19}$$

IV. Shu–Osher Representation of HB*ks*

Now we extend the Shu–Osher representation to our methods so that we can rewrite our *s*-stage HB*ks* methods in formulae (8)–(9) as convex combinations of FE method [13] to show that they will preserve the monotonicity property (5).

Firstly, let

$$\lambda_{i\ell}, \lambda_{s+1,\ell} \ge 0, \quad \sum_{\ell=1}^{i-1} \lambda_{i\ell} = 1, \quad i = 3,4,\dots,s, \text{ and } \sum_{\ell=1}^{s} \lambda_{s+1,\ell} = 1. \tag{20}$$

Then formulae (8) and (9) become

$$Y_i = \left[\sum_{\ell=1}^{i-1} \lambda_{i\ell}\right] \alpha_{i0} y_n + \sum_{j=1}^{k-1} \alpha_{ij} y_{n-j} + \Delta t \left[\sum_{j=1}^{i-1} a_{ij} F_j + \sum_{j=1}^{k-1} \beta_{ij} f_{n-j}\right], \quad i = 3, 4, \ldots, s, \qquad (21)$$

$$y_{n+1} = \left[\sum_{\ell=1}^{s} \lambda_{s+1,\ell}\right] \alpha_0 y_n + \sum_{j=1}^{k-1} \alpha_j y_{n-j} + \Delta t \left[\sum_{j=1}^{s} b_j F_j + \sum_{j=1}^{k-1} \beta_j f_{n-j}\right]. \qquad (22)$$

Secondly, express the term y_n in formulae (8) as a function of Y_i,

$$y_n = \frac{1}{\alpha_{i0}} \left\{ Y_i - \sum_{j=1}^{k-1} \alpha_{ij} y_{n-j} - \Delta t \left[\sum_{j=1}^{i-1} a_{ij} F_j + \sum_{j=1}^{k-1} \beta_{ij} f_{n-j}\right]\right\}, \quad i = 2, 3, \ldots, s, \qquad (23)$$

To avoid confusion, we replace the index *i* by *m* in formulae (23) to obtain

$$y_n = \frac{1}{\alpha_{m0}} \left\{ Y_m - \sum_{j=1}^{k-1} \alpha_{mj} y_{n-j} - \Delta t \left[\sum_{j=1}^{m-1} a_{mj} F_j + \sum_{j=1}^{k-1} \beta_{mj} f_{n-j}\right]\right\}, \quad i = 2, 3, \ldots, s, \qquad (24)$$

Thirdly, for $i = 3, 4, \ldots, s$ and $l = 1, 2, \ldots i-1$, we substitute (24) into the terms $\lambda_{i\ell}\alpha_{i0}y_n$ in (21) with $m = l$. Similarly, we replace y_n in terms of $\lambda_{s+1,\ell}\alpha_0 y_n$ in (22) by (24) with $m = l$. If we set

$$\begin{aligned} e_{ij} &= \lambda_{ij}\alpha_{i0}/\alpha_{j0}, && j = 2, 3, \ldots, i-1, \quad i = 3, 4, \ldots, s, \\ a_{i1} &= \beta_{i0}, && i = 2, 3, \ldots, s, \\ e_j &= \lambda_{s+1,j}\alpha_0/\alpha_{j0}, && j = 2, 3, \ldots, s, \\ b_1 &= \beta_0, \end{aligned}$$

then after some calculations, formulae (8) with i = 2, (21) and (22) become:

$$Y_i = \left[\sum_{j=0}^{k-1} A_{ij} y_{n-j} + \Delta t \sum_{j=0}^{k-1} B_{ij} f_{n-j}\right] + \left[\sum_{j=2}^{i-1} e_{ij} Y_j + \Delta t \sum_{j=2}^{i-1} g_{ij} F_j\right], \quad i = 2, 3, \ldots, s, \qquad (25)$$

$$y_{n+1} = \left[\sum_{j=0}^{k-1} A_j y_{n-j} + \Delta t \sum_{j=0}^{k-1} B_j f_{n-j}\right] + \left[\sum_{j=2}^{s} e_j Y_j + \Delta t \sum_{j=2}^{s} g_j F_j\right], \qquad (26)$$

where the coefficients are

$$\begin{aligned} A_{ij} &= \alpha_{i,j} - \sum_{\ell=2}^{i-1} e_{i\ell}\alpha_{\ell j}, \quad j = 0, 1, \ldots, k-1, \ i = 2, 3, \ldots, s, \\ A_j &= \alpha_j - \sum_{\ell=2}^{s} e_\ell \alpha_{\ell j}, \quad j = 0, 1, \ldots, k-1, \\ B_{ij} &= \beta_{i,j} - \sum_{\ell=2}^{i-1} e_{i\ell}\beta_{\ell j}, \quad j = 0, 1, \ldots, k-1, \ i = 2, 3, \ldots, s, \\ B_j &= \beta_j - \sum_{\ell=2}^{s} e_\ell \beta_{\ell j}, \quad j = 0, 1, \ldots, k-1, \\ g_{ij} &= a_{ij} - \sum_{\ell=j+1}^{i-1} e_{i\ell} a_{\ell j}, \ i = 3, 4, \ldots, s, \ j = 2, 3, \ldots, i-1, \\ g_j &= b_j - \sum_{\ell=j+1}^{s} e_\ell a_{\ell j}, \quad j = 2, 3, \ldots, s, \end{aligned}$$

Since it is assumed that all the coefficients in the Shu–Osher form are nonnegative, one can easily verify that

$$\sum_{j=0}^{k-1} A_{ij} + \sum_{j=2}^{i-1} e_{ij} = 1, \ i = 2, 3, \ldots, s, \quad \sum_{j=0}^{k-1} A_j + \sum_{j=2}^{s} e_j = 1.$$

Moreover, by convexity, the HB*ks* methods can be decomposed into a combination of FE method with new step sizes:

- In (25) for $i = 2, 3, \ldots, s,$ the step sizes are $\frac{B_{ij}}{A_{ij}}\Delta t,$ $j = 0, \ldots, k-1,$ and $\frac{g_{ij}}{e_{ij}}\Delta t,$ $j = 2, 3, \ldots, i-1,$ for the first and second bracketed terms, respectively.
- In (26), FE terms have step sizes $\frac{B_j}{A_j}\Delta t,\ j = 0, \ldots, k-1,$ and $\frac{g_j}{e_j}\Delta t,\ j = 2, 3, \ldots, s,$ in the first and second brackets, respectively.

Thus, we can conclude:

Theorem 1: If the FE method is SSP under the time-step restriction $\Delta t \le \Delta t_{\mathrm{FE}},$ then the *k*-step, *s*-stage HB*ks* (25)–(26) preserve the monotonicity (5), that is the method is SSP under the restriction

$$\Delta t \le c(\mathrm{HB}ks)\Delta t_{\mathrm{FE}},$$

where the SSP coefficient *c*(HB*ks*) is the minimum of the following quotients:

$$\min_{j=0,1,\ldots,k-1} \left\{\frac{A_j}{B_j}\right\}, \quad \min_{j=0,1,\ldots,k-1} \left\{\frac{A_{ij}}{B_{ij}}\right\}, \ i = 2, 3, \ldots, s, \qquad (27)$$

$$\min_{j=2,3,\ldots,s} \left\{\frac{e_j}{g_j}\right\}, \quad \min_{j=2,3,\ldots,i-1} \left\{\frac{e_{ij}}{g_{ij}}\right\}, \ i = 3, 4, \ldots, s, \qquad (28)$$

with the convention that $a/0 = +\infty$ and under the assumption that all coefficients of (25)–(26) are nonnegative.

This result is the following straightforward extension of the result presented in [5], [6] and the idea for the proof is in [14].

V. Construction of Optimal HB*ks*

In order to start, we search for an optimal k-step, s-stage HB scheme by maximizing its SSP coefficients according to Theorem 1. This means we seek the global maximum of the nonlinear programming problem

$$\max_{A_{ij},B_{ij},e_{ij},g_{ij},A_j,B_j,e_j,g_j} c(\mathrm{HB}ks), \tag{29}$$

where all the numbers in all pairs

$$\{A_j,B_j\},\quad j=0,1,\dots,k-1,$$
$$\{A_{ij},B_{ij}\},\quad i=2,3,\dots,s,\quad j=0,1,\dots,k-1,$$
$$\{e_j,g_j\},\quad j=2,3,\dots,s,$$
$$\{e_{ij},g_{ij}\},\quad i=3,4,\dots,s,\quad j=2,3,\dots,i-1,$$

are nonnegative real numbers. Note that if the coefficients in a pair are zero, it is defined as NaN (in MATLAB), and hence it is not included in the minimization process. Furthermore, the objective function (29) is subject to

- the convex combination constraints (20),
- the simplifying assumptions (11) for HB*ks*,
- the order conditions (12) to (18) for HB*ks*,
- the conditions on the abscissae c_i: $c_1=0,\ 0\le c_i\le 1$, $i = 2, 3, \dots, s$.

VI. Comparing Effective SSP Coefficients

A. Effective SSP Coefficients and Percentage Efficiency Gain

When the time step is limited by the monotonicity property, the computational efficiency of the method can be measured by the *effective SSP coefficients*, which is very helpful to make a fair comparison between two methods of the same order.

Definition 1: (See [12]) The *effective SSP coefficients* of an SSP method M, $c_{\mathrm{eff}}(M)$ is defined by

$$c_{\mathrm{eff}}(M)=\frac{c(M)}{\ell}, \tag{30}$$

where ℓ is the number of function evaluations of method M used per time step and $c(M)$ is the SSP coefficient of M.
For instance, in this paper, ℓ is also the number of the stages for HB*ks* or RK*s* method and $\ell = 2$ for HM*k*. By definition, $c_{\mathrm{eff}}(FE)=1$.

Definition 2: (See [15]) The *percentage efficiency gain* (PEG) of the effective SSP coefficients $c_{\mathrm{eff}}(\mathrm{M2})$ of method 2 over $c_{\mathrm{eff}}(\mathrm{M1})$ of method 1 is evaluated by

$$\mathrm{PEG}(c_{\mathrm{eff}}(\mathrm{M2}),c_{\mathrm{eff}}(\mathrm{M1}))=\frac{c_{\mathrm{eff}}(\mathrm{M2})-c_{\mathrm{eff}}(\mathrm{M1})}{c_{\mathrm{eff}}(\mathrm{M1})}. \tag{31}$$

B. List of Known SSP Methods of Order 4

We now list some known SSP methods of order 4 in the literature so that we can compare them to HB*ks*. Shu and Osher [13] constructed a series of second- to fifth-order SSP RK, several of which are downwinded ones. Ruuth and Spiteri [11] showed that there is no s-stage SSP RK method of order $p > 4$ with nonnegative coefficients. Moreover, they also proved that there is no four-stage explicit SSP RK method of order 4 with positive SSP coefficients (see more in [12]). However, in [16], they constructed s-stage RK method of order 4 with nonnegative SSP coefficients for $k = 5,\dots,8$. The 10-stage RK method of order 4 implemented with two registers and a great effective SSP coefficient was found by Ketcheson.

TABLE I
$c_{\mathrm{EFF}}(\mathrm{HB}ks)$ FOR $k=2,3,4,5$ AND $s=4,5,\dots,10$.

s	HB2s	HB3s	HB4s	HB5s
4	0.398	0.461	0.483	0.494
5	0.452	0.504	0.508	
6	0.488	0.512	0.514	0.515
7	0.532	0.534	0.536	
8	0.553	0.5538	0.5544	
9	0.586	0.587		
10	0.610	0.614		

- RK5 with $c(\mathrm{RK5})=1.508$ and $c_{\mathrm{eff}}(\mathrm{RK5})=0.302$,
- RK6 with $c(\mathrm{RK6})=2.295$ and $c_{\mathrm{eff}}(\mathrm{RK6})=0.382$,
- RK7 with $c(\mathrm{RK7})=3.321$ and $c_{\mathrm{eff}}(\mathrm{RK7})=0.474$,
- RK8 with $c(\mathrm{RK8})=4.146$ and $c_{\mathrm{eff}}(\mathrm{RK8})=0.518$,
- RK9 with $c(\mathrm{RK9})=4.869$ and $c_{\mathrm{eff}}(\mathrm{RK9})=0.541$,
- RK10 with $c(\mathrm{RK10})=6.000$ and $c_{\mathrm{eff}}(\mathrm{RK10})=0.600$.

For HM method, in [6], Huang found the following k-step hybrid methods of order 4 with nonnegative coefficients:

- HM3 with $c(\mathrm{HM3})=0.494$ and $c_{\mathrm{eff}}(\mathrm{HM3})=0.247$,
- HM4 with $c(\mathrm{HM4})=0.682$ and $c_{\mathrm{eff}}(\mathrm{HM4})=0.341$,
- HM5 with $c(\mathrm{HM5})=0.793$ and $c_{\mathrm{eff}}(\mathrm{HM5})=0.396$,
- HM6 with $c(\mathrm{HM6})=0.879$ and $c_{\mathrm{eff}}(\mathrm{HM6})=0.439$,
- HM7 with $c(\mathrm{HM7})=0.938$ and $c_{\mathrm{eff}}(\mathrm{HM7})=0.469$.

Gottlieb, Shu and Tadmor [5] proved that there is no two-step hybrid multistep RK method of order 4. However, in our research, we have constructed 2-step, s-stage HB of order 4 with non-negative coefficients for $s = 4,\dots,10$.

In the case of general linear methods, Constantinescu and Sandu discovered GL24, GL34 and GL44 (see more in [2]).

- GL24 with $c(\mathrm{GL24})=1.590$ and $c_{\mathrm{eff}}(\mathrm{GL24})=0.398$,
- GL34 with $c(\mathrm{GL34})=1.840$ and $c_{\mathrm{eff}}(\mathrm{GL34})=0.461$,
- GL44 with $c(\mathrm{GL44})=1.930$ and $c_{\mathrm{eff}}(\mathrm{GL44})=0.483$.

C. Comparing Our New HBks with the Above-Mentioned SSP Methods

The c_{eff} coefficients and the formulae of HB*ks* methods are listed in Table I and in Appendix A.

Remark 1: From Table I, we can observe that:

- For a fixed stage, the effective SSP coefficients increase as the number of steps k increases because the set of feasible solutions of Problem (29) is larger.
- For a fixed step, especially in the case of order 4, the effective SSP coefficients increase up to ten stages when the number of stages increases. However, it does not happen for all order. In fact, when the set of feasible solutions of Problem (29) is larger, c(HB) increases, but if the difference between c(HB) at s-stage and (s + 1)- stage is smaller than c_{eff} (HB) at s-stage, then c_{eff} at (s + 1)-stage decreases.

TABLE II

LOWER RIGHT BLOCK: PEG(c_{eff}) OF HB*ks* OVER HM*k* AND RK*s*

			HM3	HM4	HM5	HM6	RK5	RK6	RK7	RK8	RK9	RK10
	$c \backslash c$		0.494	0.682	0.793	0.879	1.508	2.295	3.321	4.146	4.869	6.000
		$c_{\text{eff}} \backslash c_{\text{eff}}$	0.247	0.341	0.396	0.439	0.302	0.382	0.474	0.518	0.541	0.600
HB54	1.979	0.494	80%	45%	25%	13%	64%	29%	4%	-5%	-9%	
HB35	2.520	0.504	104%	48%	27%	15%	67%	32%	6%	-3%	-7%	
HB56	3.093	0.515	109%	51%	30%	17%	71%	35%	9%	-1%		
HB37	3.741	0.534		57%	35%	22%	77%	40%	13%	3%	-1%	
HB28	4.424	0.553			40%	26%	83%	45%	17%	7%	2%	-8%
HB39	5.279	0.587				34%	94%	54%	24%	13%	9%	-2%
HB210	6.102	0.610	147%	79%	54%	39%	102%	60%	29%	18%	13%	2%

In Table II, we compare the efficiency of our methods with the efficiency of other methods.

Remark 2:

- At first glance, we can see that our best HB*ks* method is HB310. In fact, its c_{eff} is greater than all of the hybrid methods and RK methods, especially RK10. While HB210 has the same c_{eff} with the optimal explicit two-step RK method of order 4, HB310 is slightly better (See more in [4]).
- All of our new methods (except for one method HB24) have the greater c_{eff} than those of hybrid methods listed in Table II.
- As shown in Table I and II, RK5 and RK6 have the smaller c_{eff} than those of HB*ks* methods.
- In comparison with RK*s* for s = 7,..., 10, the c_{eff} of HB*ks* with multistep are larger for the same stage.
- GL methods and HB methods have the same SSP coefficients and c_{eff} for the same number of steps and stages.

Therefore, our investigation, to obtain HB*ks* methods of order 4 by combining linear *k*-step methods and 4- to 10-stage RK methods of order 4, shows that these new SSP methods have good SSP coefficients.

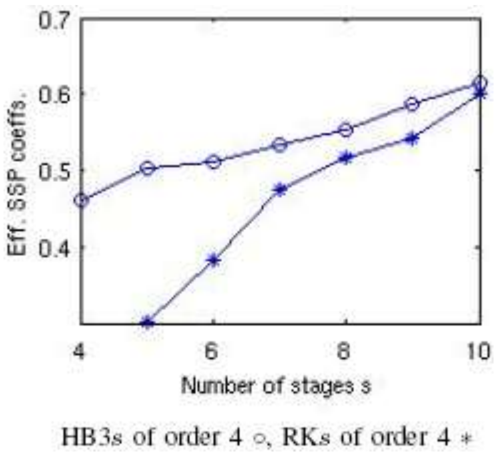

Fig. 1. Effective SSP coefficients versus number of stages, *s*, of 3-step HB3*s* methods of order 4 and RK*s* methods of order 4, including Ketcheson's RK10

Fig. 1 compares HB3*s* and RK*s* of order 4 on the basis of their effective SSP coefficients as functions of their number of stages. Clearly, the new methods, generally, have larger effective SSP coefficients, especially when the number of stages of both methods is small.

VII. NUMERICAL RESULTS

In this section, we present some numerical results to confirm the validity of our new optimal schemes on two problems of inviscid Burgers' equation with different initial conditions: the first problem is with unit downstep initial condition and the second is with a square-wave initial condition.

From now on we will use the total variation semi-norm $\|y_n\| = TV(y_n)$ where

$$TV(y_n) = \sum_j |y_{n,j+1} - y_{n,j}|, \tag{32}$$

and say that a method is total variation diminishing (TVD) if

$$TV(y_{n+1}) \le TV(y_n). \tag{33}$$

The following two definitions will help us compare different methods with different computational costs more easily and fairly (See more in [6]).

Definition 3: The *largest effective CFL number* of method *M* denoted by $n_{\text{eff}}(M)$, for error ϵ,

$$|\text{TV}(u(x,t_{\text{final}})) - \text{TV}(u(x,t_0))| \le \epsilon, \tag{34}$$

is defined by

$$n_{\text{eff}}(M) = \max_{\Delta t}\left\{\frac{\Delta t}{\Delta x}\frac{1}{\ell}\right\}, \tag{35}$$

where ℓ is the number of function evaluations per step of the method.

Then $\max \Delta t_{\text{num}} = \ell \Delta x n_{\text{eff}}(M)$ is called the *maximum numerical stepsize*. We let $\max\{\Delta t_{\text{theor}}\}$ be the *maximum theoretical time step* taken as

$$\max \Delta t_{\text{theor}} = c(M)\Delta t_{\text{FE}} \tag{36}$$

and $R_{\text{n/t}}$ be the ratio of the maximum numerical and theoretical stepsizes

$$R_{\text{n/t}} = \frac{\max \Delta t_{\text{num}}}{\max \Delta t_{\text{theor}}}. \tag{37}$$

Definition 4: The *percentage efficiency gain*, PEG(n_{eff}), of n_{eff} of method 2 over method 1, is defined by

$$\mathrm{PEG}(n_{\text{eff}}) = \frac{n_{\text{eff}}(\mathrm{M2}) - n_{\text{eff}}(\mathrm{M1})}{n_{\text{eff}}(\mathrm{M1})}. \tag{38}$$

To apply the HB schemes, we need some starting values computed by using the optimal RK54 scheme [11] with a small initial step size $h \approx 1.0\text{e-}04$.

TABLE III

n_{eff} (HB*KS*) FOR HB*KS* APPLIED TO PROBLEM 1

s	HB2s	HB3s	HB4s	HB5s
4	0.390	0.365	0.380	0.380
5	0.385	0.405	0.395	
6	0.430	0.440	0.420	0.421
7	0.421	0.426	0.421	
8	0.370	0.375	0.377	
9	0.360	0.365		
10	0.350	0.360		

A. Burgers' equation with a unit downstep initial condition

As in Huang [6], we consider the following problem.

Problem 1: Burgers' equation with a unit downstep initial condition,

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[\frac{1}{2}\,u(x,t)^2\right]=0,$$

$$u(x,0)=\begin{cases}1, & -1\le x<0,\\ 0, & 0<x\le 1.\end{cases} \tag{39}$$

and boundary condition

$$u(-1,t)=1\,,\ t\ge 0.$$

First, we discretize the spatial derivative of the flux function $f(u)=u(x,t)^2/2$ by the weighted essentially non-oscillatory finite difference scheme of order 5 (WENO5) of Jiang and Shu [8] with the spatial step size $\Delta x = 1/150$ to obtain the semi-discrete system

$$\frac{d}{dt}u_j(t) = -\frac{1}{\Delta x}\left[f_{j+(1/2)} - f_{j-(1/2)}\right], \tag{40}$$

where $u_j(t) \approx u(x_j,t)$ with $x_j = j\Delta x$, j = -150, -149, …., 149, 150, and $f_{j+(1/2)}$ is the numerical flux, which typically is a Lipschitz continuous function of several neighbouring values $u_j(t)$ (see [8] for details). Now a time discretization can be applied to (40). For Problem 1, we consider the total variation norm of the numerical solution at $t_{\text{final}} = 1.8$ and take Δt sufficiently small such that (34) holds with $\epsilon = 5.0\text{e-}02$.

The numerical results show that the FE method satisfies the TVD property (33) under time step restriction

$$\Delta t \le \Delta t_{\mathrm{FE}} = 0.325\Delta x. \tag{41}$$

The $n_{\text{eff}}(\mathrm{HB}ks)$, for $s=4,5,\ldots,10$, as a function of s for this problem are listed in Table III.

The largest effective CFL numbers n_{eff} of several HB*ks*, and HM*k* for k = 3,…., 6 and RK10 applied to Problem 1 are listed and compared in Table IV.

TABLE IV

LOWER RIGHT BLOCK: PEG(n_{eff}) OF HB*KS* OVER HM*K* AND RK10 AND RATIO $R_{\text{N/T}}$=MAX ΔT_{NUM}/MAX ΔT_{THEOR} FOR HB*KS*, HM*K* AND RK10 APPLIED TO PROBLEM 1

			HM3	HM4	HM5	HM6	RK10
	$n_{\text{eff}} \backslash n_{\text{eff}}$		0.214	0.264	0.329	0.274	0.346
		$R_{\text{n/t}} \backslash R_{\text{n/t}}$	2.666	2.382	2.553	1.918	1.774
HB24	0.390	3.013	82%	48%	19%	42%	13%
HB34	0.365	2.437	71%	38%	11%	33%	5%
HB25	0.385	2.623	80%	46%	17%	41%	11%
HB36	0.440	2.647	101%	63%	31%	57%	24%
HB47	0.421	2.434	97%	60%	28%	54%	22%
HB48	0.377	2.092	76%	43%	15%	38%	9%
HB29	0.360	1.891	68%	36%	9%	31%	4%
HB210	0.350	1.765	64%	33%	6%	28%	1%

The lower right block lists the $\mathrm{PEG}(c_{\text{eff}})$ of HB*ks* over the other methods on hand.

Remark 3:

- HB*ks* have larger n_{eff} than HM methods and RK10, thus every *percentage efficiency gain* is positive.
- Among HB schemes, although HB210 has the largest n_{eff}=0.610 (see Table II), it has the smallest n_{eff} when applied to Problem 1.
- The HB36 has the greatest n_{eff} among the HB*ks* methods listed in Table III.

B. Burgers' Equation with a Square-Wave Initial Condition

In the next comparison, we use the fourth case of Laney's five test problems [9, p. 312].

Problem 2: Burgers' equation with a square-wave initial condition,

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[\frac{1}{2}\,u(x,t)^2\right]=0,$$

$$u(x,0)=\begin{cases}1, & |x|\le \frac{1}{3},\\ 0, & \frac{1}{3}<|x|\le 1.\end{cases} \tag{42}$$

and boundary condition $u(-1,t)=0$ for $t\ge 0$.

We use the same procedure as in Problem 1; however, now t_{final}=0.6 and n_{eff} (FE)=0.183 instead of 0.325.

TABLE V

n_{eff} (HB*KS*) FOR HB*KS* APPLIED TO PROBLEM 2

s	HB2s	HB3s	HB4s	HB5s
4	0.405	0.350	0.375	0.375
5	0.410	0.405	0.395	
6	0.415	0.435	0.451	
7	0.446	0.441	0.450	
8	0.400	0.410	0.410	
9	0.400	0.405		
10	0.390	0.390		

In Table V, we list n_{eff}(HB*ks*) for s = 4, 5, …, 10 and k = 2, …, 5 for HB*ks* applied to Problem 2.

The n_{eff} of HB*ks*, HM*k* and RK10 applied to Problem 2 are listed in Table VI, respectively. The lower right block shows PEG(c_{eff}) of HB*ks* over the other methods on hand.

TABLE VI

LOWER RIGHT BLOCK: PEG(n_{eff}) OF HB*KS* OVER HM*K* AND RK10 AND RATIO $R_{\text{N/T}}$=MAX ΔT_{NUM}/MAX ΔT_{THEOR} FOR HB*KS*, HM*K* AND RK10 APPLIED TO PROBLEM 2

			HM3	HM4	HM5	HM6	RK10
	$n_{\text{eff}} \backslash n_{\text{eff}}$		0.210	0.280	0.310	0.204	0.371
		$R_{n/t} \backslash R_{n/t}$	4.638	4.480	4.265	2.532	3.373
HB24	0.405	5.548	93%	45%	31%	99%	9%
HB34	0.350	4.144	67%	25%	13%	72%	-0.1%
HB25	0.410	4.953	95%	46%	32%	101%	11%
HB36	0.435	4.640	107%	55%	40%	113%	17%
HB47	0.450	4.584	114%	61%	45%	121%	21%
HB28	0.400	3.946	91%	43%	29%	96%	8%
HB210	0.390	3.487	86%	39%	26%	91%	5%

Remark 4:

- For Problem 2, the $n_{\text{eff}}(\text{HB}ks)$ listed in Table V, except for HB34 = 0.350, are larger than the n_{eff} of HM*k* and RK10 listed in Table VI.
- Although, according to Table I, $c_{\text{eff}}(\text{HB25}) = 0.452$ is smaller than $c_{\text{eff}}(\text{HB38}) = 05538$ and $c_{\text{eff}}(\text{HB48}) =$ 0.5544, it is seem in Table V that they have the same $n_{\text{eff}} = 0.410$ when applied to Problem 2. Similarly, $c_{\text{eff}}(\text{HB35}) = 0.504 < c_{\text{eff}}(\text{HB39}) = 0.587,$ but have the same $n_{\text{eff}} = 0495$ when applied to Problem 2.
- Among the HB*ks* methods, HB36 and HB46 obtain the highest n_{eff} for Problem 1 and 2, respectively (see Tables III and V).

VIII. CONCLUSION

In our research, we extend the Shu–Osher representation for Runge–Kutta methods [13] to a Shu–Osher representation of HB methods. Moreover, under this representation, new series of optimal, explicit, *k*-step, *s*-stage, SSP Hermite–Birkhoff methods, HB*ks*, of order 4 for *s* = 4, 5,…., 10, with nonnegative coefficients are obtained by combining linear *k*-step methods with 4- to 10-stage Runge–Kutta methods of order 4. Although some methods, such as HM, GL, and RK methods of the same order, have been found in the literature, HB methods have better SSP coefficients and larger maximum effective CFL numbers when tested on Burgers' equation.

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APPENDIX

This appendix lists the Shu–Osher representation of the HB*ks* methods considered in this paper with their $c(\text{HB}ks)$, $c_{\text{eff}}(\text{HBks})$ and abscissa vector $\sigma = [c_1, c_2, \ldots]$.

HB24. Here $c = 1.593$, $c_{\text{eff}} = 0.398$, and
$\sigma = [0, 0.41402182360075634, 0.51006789128486130, 0.95950236308462156]^T$.

$$
\begin{aligned}
Y_2 &= 1.3137364256438025\,e\text{-}01 y_{n-1} + 8.6862635743561967\,e\text{-}01 y_n \\
&\quad + 5.4539546616513657\,e\text{-}01 \Delta t f_n, \\
Y_3 &= 3.7611628586354845\,e\text{-}01 y_{n-1} + 2.3615691062661964\,e\text{-}01 \Delta t f_{n-1} \\
&\quad + 6.2388371413645161\,e\text{-}01 Y_2 + 3.9172579348020337\,e\text{-}01 \Delta t F_2, \\
Y_4 &= 6.0575318286657687\,e\text{-}02 y_{n-1} + 4.3008189202816044\,e\text{-}02 y_n \\
&\quad + 8.9641649251052635\,e\text{-}01 Y_3 + 5.6284441132346308\,e\text{-}01 \Delta t F_3, \\
y_{n+1} &= 4.4524211403575709\,e\text{-}02 y_{n-1} + 3.8295387371954609\,e\text{-}01 y_n \\
&\quad + 2.4045011377921155\,e\text{-}01 \Delta t f_n + 1.6774221781036905\,e\text{-}01 Y_2 \\
&\quad + 9.2083912086026398\,e\text{-}02 \Delta t F_2 + 4.0477969706650918\,e\text{-}01 Y_4 \\
&\quad + 2.5415417076166058\,e\text{-}01 \Delta t F_4.
\end{aligned}
$$

HB34. Here $c = 1.843$, $c_{\text{eff}} = 0.461$, and
$\sigma = [0, 0.30622814192704767, 0.55930144311839380, 0.88515753145046949]^T$.

$$
\begin{aligned}
Y_2 &= 1.8162651927653470\,e\text{-}01 y_{n-1} + 8.1837348072346527\,e\text{-}01 y_n \\
&\quad + 4.3925422184221165\,e\text{-}02 \Delta t f_{n-1} + 4.4392923901936121\,e\text{-}01 \Delta t f_n, \\
Y_3 &= 2.2153848827415687\,e\text{-}01 y_{n-1} + 1.2017424174852812\,e\text{-}01 \Delta t f_{n-1} \\
&\quad + 7.7846151172584321\,e\text{-}01 Y_2 + 4.2227886734649689\,e\text{-}01 \Delta t F_2, \\
Y_4 &= 6.7530828539879564\,e\text{-}04 y_{n-2} + 1.3756312556575984\,e\text{-}01 y_{n-1} \\
&\quad + 7.4621545159975711\,e\text{-}02 \Delta t f_{n-1} + 8.6176156614884136\,e\text{-}01 Y_3 \\
&\quad + 4.6746524085603702\,e\text{-}01 \Delta t F_3, \\
y_{n+1} &= 1.0945244030080964\,e\text{-}02 y_{n-2} + 4.4070641867034777\,e\text{-}01 y_n \\
&\quad + 2.3906256700586392\,e\text{-}01 \Delta t f_n + 5.4834833729957133\,e\text{-}01 Y_4 \\
&\quad + 2.9745326043523995\,e\text{-}01 \Delta t F_4.
\end{aligned}
$$

HB44. Here $c = 1.932$, $c_{\text{eff}} = 0.483$, and
$\sigma = [0, 0.29550672532505878, 0.55487098371182864, 0.87509757670763788]^T$.

$$
\begin{aligned}
Y_2 &= 4.7524774508095377\,e\text{-}02 y_{n-2} + 1.2685090981790423\,e\text{-}01 y_{n-1} \\
&\quad + 8.2562431567400041\,e\text{-}01 y_n + 2.4589659756032358\,e\text{-}02 \Delta t f_{n-2} \\
&\quad + 6.5633572056908576\,e\text{-}02 \Delta t f_{n-1} + 4.2718395234621287\,e\text{-}01 \Delta t f_n,
\end{aligned}
$$

$$
\begin{aligned}
Y_3 &= 1.9918300748893264\,\text{e-}01 y_{n-1} + 1.0305871903720029\,\text{e-}01 \Delta t f_{n-1}\\
&\quad + 8.0081699251106719\,\text{e-}01 Y_2 + 4.1434846512195339\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 1.2681476034277145\,\text{e-}01 y_{n-1} + 6.5614868058771342\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 8.7318523965722883\,\text{e-}01 Y_3 + 4.5179231610038256\,\text{e-}01 \Delta t F_3,\\
y_{n+1} &= 3.2258872667719520\,\text{e-}03 y_{n-3} + 1.4711725267828712\,\text{e-}02 y_{n-1}\\
&\quad + 4.0081143477759190\,\text{e-}01 y_n + 7.6119523449503096\,\text{e-}03 \Delta t f_{n-1}\\
&\quad + 2.0738271584706414\,\text{e-}01 \Delta t f_n + 5.8125095268780758\,\text{e-}01 Y_4\\
&\quad + 3.0074341872002397\,\text{e-}01 \Delta t F_4.
\end{aligned}
$$

HB25. Here $c = 2.258$, $c_{\text{eff}} = 0.452$, and
$\sigma = [0, 0.43353487829608683, 0.49212970792955379, 0.51952905303075192, 0.95306393132683886]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 4.3353487829608683\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 2.6154930329164866\,\text{e-}01 y_{n-1} + 1.1339074537097120\,\text{e-}01 \Delta t f_{n-1}\\
&\quad + 7.3845069670835128\,\text{e-}01 Y_2 + 3.2014413292511562\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 4.1933308968746225\,\text{e-}01 y_n + 1.0527799621252554\,\text{e-}02 y_{n-1}\\
&\quad + 2.2992686629013520\,\text{e-}03 \Delta t f_{n-1} + 5.7013911069128531\,\text{e-}01 Y_3\\
&\quad + 2.4717518996538551\,\text{e-}01 \Delta t F_3,\\
Y_5 &= 1.0 Y_4 + 4.3353487829608683\,\text{e-}01 \Delta t F_4,\\
y_{n+1} &= 2.3548942789448177\,\text{e-}02 y_{n-1} + 2.1442551411227079\,\text{e-}01 y_n\\
&\quad + 9.2960939164239198\,\text{e-}02 \Delta t f_n\\
&\quad + 2.4250812182244122\,\text{e-}01 Y_2 + 1.0509003662565282\,\text{e-}01 \Delta t F_2\\
&\quad + 5.1951742127583977\,\text{e-}01 Y_4 + 2.2522892200551811\,\text{e-}01 \Delta t F_4
\end{aligned}
$$

HB35. Here $c = 2.520$, $c_{\text{eff}} = 0.504$, and
$\sigma = [0, 0.39667202894458992, 0.34733932541243928, 0.74401135435702903, 0.87211776033132926]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 3.9667202894458869\,4888\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 3.1933390462023281\,\text{e-}01 y_{n-1} + 1.2667082785650549\,\text{e-}01 \Delta t f_{n-1}\\
&\quad + 6.8066609537976719\,\text{e-}01 Y_2 + 2.7000120108808318\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 1.0 Y_3 + 3.9667202894458875\,\text{e-}01 \Delta t F_3,\\
Y_5 &= 3.0842709893420334\,\text{e-}01 y_n + 9.7477349638087907\,\text{e-}02 \Delta t f_n\\
&\quad + 1.9119743013194673\,\text{e-}02 Y_2 + 6.7245315805260308\,\text{e-}01 Y_4\\
&\quad + 2.6674335857492221\,\text{e-}01 \Delta t F_4,\\
y_{n+1} &= 3.7295391119407718\,\text{e-}03 y_{n-2} + 1.6897464146259084\,\text{e-}01 y_n\\
&\quad + 5.2526715235296312\,\text{e-}02 \Delta t f_n + 1.9426370492958658\,\text{e-}01 Y_2\\
&\quad + 7.4688701847753544\,\text{e-}02 \Delta t F_2\\
&\quad + 6.3303211449588204\,\text{e-}01 Y_5 + 2.5110613324416470\,\text{e-}01 \Delta t F_5.
\end{aligned}
$$

HB45. Here $c = 2.542$, $c_{\text{eff}} = 0.508$, and
$\sigma = [0, 0.39327099964963397, 0.36835269419472677, 0.7258618118940602, 0.85325549612031670]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 3.9327099964963397\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 3.0014929271455676\,\text{e-}01 y_{n-1} + 1.1804001238998434\,\text{e-}01 \Delta t f_{n-1}\\
&\quad + 6.9985070728544319\,\text{e-}01 Y_2 + 2.7523098725964962\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 2.6134988517230506\,\text{e-}02 y_{n-1} + 1.0278133060002928\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 9.7386501148276960\,\text{e-}01 Y_3 + 4.4332192978558854\,\text{e-}01 \Delta t F_3,\\
Y_5 &= 3.6628958743758816\,\text{e-}01 y_n + 1.4405107221283228\,\text{e-}01 \Delta t f_n\\
&\quad + 2.3307576925754755\,\text{e-}06 Y_2\\
&\quad + 6.3370808180471916\,\text{e-}01 Y_4 + 2.4921901081739392\,\text{e-}01 \Delta t F_4,\\
y_{n+1} &= 9.6422758663194897\,\text{e-}04 y_{n-3} + 1.5868186323471442\,\text{e-}01 y_n\\
&\quad + 3.3264667026468547\,\text{e-}02 \Delta t f_n + 1.5915027906866219\,\text{e-}01 Y_2\\
&\quad + 5.7900452442798594\,\text{e-}02 \Delta t F_2\\
&\quad + 6.8120363010999174\,\text{e-}01 Y_5 + 2.6789763257831589\,\text{e-}01 \Delta t F_5.
\end{aligned}
$$

HB26. Here $c = 2.930$, $c_{\text{eff}} = 0.488$, and
$\sigma = [0, 0.34131501751573518, 0.58493205600266185, 0.5907859011009666, 0.58032954089932975, 0.89749336077124797]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 3.4131501751573518\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 7.2837460069414606\,\text{e-}02 y_{n-1} + 2.4860518959393910\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 9.2716253993058551\,\text{e-}01 Y_2 + 3.1645449855634128\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 2.1165650044551004\,\text{e-}01 y_{n-1} + 7.2241542156878474\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 7.8834349955449001\,\text{e-}01 Y_3 + 2.6907347535885678\,\text{e-}01 \Delta t F_3\\
Y_5 &= 5.2217429737665433\,\text{e-}01 y_n + 1.7822592945537952\,\text{e-}01 \Delta t f_n\\
&\quad + 7.3255243277284984\,\text{e-}02 Y_2\\
&\quad + 4.0457045934606084\,\text{e-}01 Y_4 + 1.3808597341804979\,\text{e-}01 \Delta t F_4
\end{aligned}
$$

$$
\begin{aligned}
Y_6 &= 4.1616350378895876\,\text{e-}02 y_n + 1.4204285358513801\,\text{e-}02 \Delta t f_n\\
&\quad + 2.3181094241883033\,\text{e-}10 Y_2\\
&\quad + 9.5838364938929321\,\text{e-}01 Y_5 + 3.2711073207810087\,\text{e-}01 \Delta t F_5,\\
y_{n+1} &= 1.3882061429735554\,\text{e-}02 y_{n-1} + 1.0652255784084119\,\text{e-}01 y_n\\
&\quad + 3.6357748695267644\,\text{e-}02 \Delta t f_n\\
&\quad + 1.8790001888305066\,\text{e-}01 Y_2 + 5.6513205020659152\,\text{e-}02 \Delta t F_2\\
&\quad + 6.9169536184637270\,\text{e-}01 Y_6 + 2.3608601454414754\,\text{e-}01 \Delta t F_6.
\end{aligned}
$$

HB36. Here $c = 3.069$, $c_{\text{eff}} = 0.512$, and
$\sigma = [0, 0.32582165651383960, 0.37067798466591867, 0.69649964117975816, 0.66867140411649717, 0.82079730062345002]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 3.2582165651383960\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 2.1191789029946936\,\text{e-}01 y_{n-1} + 6.9047438062291244\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 7.8808210970053061\,\text{e-}01 Y_2 + 2.5677421845154841\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 1.0\,\text{e-}01 Y_3 + 3.2582165651383960\,\text{e-}01 \Delta t F_3\\
Y_5 &= 5.0766128220635420\,\text{e-}01 y_n + 1.6540703991641412\,\text{e-}01 \Delta t f_n\\
&\quad + 9.1877548437183622\,\text{e-}05 Y_2\\
&\quad + 4.9224684024520876\,\text{e-}01 Y_4 + 1.6038468090239727\,\text{e-}01 \Delta t F_4\\
Y_6 &= 2.5976250331144174\,\text{e-}01 y_n + 8.4636249129115701\,\text{e-}02 \Delta t f_n\\
&\quad + 3.2614058591337942\,\text{e-}09 Y_2\\
&\quad + 7.4023749342715239\,\text{e-}01 Y_5 + 2.4118540632208724\,\text{e-}01 \Delta t F_5\\
y_{n+1} &= 1.7848294943323458\,\text{e-}03 y_{n-2} + 1.1378607462930097\,\text{e-}01 y_n\\
&\quad + 2.6029211669753471\,\text{e-}02 \Delta t f_n\\
&\quad + 6.5472471465212512\,\text{e-}02 Y_2 + 1.7176907587067886\,\text{e-}02 \Delta t F_2\\
&\quad + 8.1895662441115424\,\text{e-}01 Y_6 + 2.6683380397862461\,\text{e-}01 \Delta t F_6.
\end{aligned}
$$

HB46. Here $c = 3.083$, $c_{\text{eff}} = 0.514$, and
$\sigma = [0, 0.23240140560551406, 0.41205152977237552, 0.64391403354100463, 0.70724957298715663, 0.79393220270642573]^T$.

$$
\begin{aligned}
Y_2 &= 9.1906591765132897\,\text{e-}02 y_{n-1} + 9.0809340823486706\,\text{e-}01 y_n\\
&\quad + 2.9806042720511843\,\text{e-}02 \Delta t f_{n-1} + 2.9450195465013512\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 1.1737886093428368\,\text{e-}01 y_{n-1} + 3.8066903323245214\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 8.8262113906571626\,\text{e-}01 Y_2 + 2.8624109404740172\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 1.2554396350474908\,\text{e-}01 y_n\\
&\quad + 8.7445603649525105\,\text{e-}01 Y_3 + 2.8359308598444821\,\text{e-}01 \Delta t F_3\\
Y_5 &= 4.0529083748859801\,\text{e-}01 y_n + 1.3143905985859958\,\text{e-}01 \Delta t f_n\\
&\quad + 5.9470916251140182\,\text{e-}01 Y_4 + 1.9286893751204742\,\text{e-}01 \Delta t F_4\\
Y_6 &= 3.3598516948937474\,\text{e-}01 y_n + 1.0896267746333649\,\text{e-}01 \Delta t f_n\\
&\quad + 6.6401483051062526\,\text{e-}01 Y_5 + 2.1534531990731043\,\text{e-}01 \Delta t F_5\\
y_{n+1} &= 4.8476621034632518\,\text{e-}04 y_{n-3} + 1.3133716447259133\,\text{e-}01 y_n\\
&\quad + 3.0622680695400140\,\text{e-}02 \Delta t f_n\\
&\quad + 8.6817806931706254\,\text{e-}01 Y_6 + 2.8155709102133125\,\text{e-}01 \Delta t F_6.
\end{aligned}
$$

HB27. Here $c(\text{HB27}) = 3.726$, $c_{\text{eff}}(\text{HB24}) = 0.532$, and
$\sigma = [0, 0.26841980744987709, 0.36782834082903565, 0.63624814827891274, 0.67218202479193823, 0.60626599422072069, 0.87468580167059773]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 2.6841980744987709\,\text{e-}01 \Delta t f_n,\\
Y_3 &= 1.3324553359862062\,\text{e-}01 y_{n-1} + 3.5765740472097872\,\text{e-}02 \Delta t f_{n-1}\\
&\quad + 8.6675446640137932\,\text{e-}01 Y_2 + 2.3265406697777921\,\text{e-}01 \Delta t F_2,\\
Y_4 &= 1.00\,\text{e+}00 Y_3 + 2.6841980744987709\,\text{e-}01 \Delta t F_3,\\
Y_5 &= 3.6364127191257684\,\text{e-}01 y_n + 9.7608520187602299\,\text{e-}02 \Delta t f_n\\
&\quad + 1.7600759196159160\,\text{e-}013 Y_2 + 6.3459865216780720\,\text{e-}01 Y_4\\
&\quad + 1.7033884802283433\,\text{e-}01 \Delta t F_4,\\
Y_6 &= 4.9707943453901537\,\text{e-}01 y_n + 1.3342596610625629\,\text{e-}01 \Delta t f_n\\
&\quad + 3.0940611702902734\,\text{e-}04 Y_2 + 5.0261115934395562\,\text{e-}01 Y_5\\
&\quad + 1.3491079061326405\,\text{e-}01 \Delta t F_5,\\
Y_7 &= +1.00\,\text{e+}00 Y_6 + 2.6841980744987703\,\text{e-}01 \Delta t F_6,\\
y_{n+1} &= 7.4249746253942581\,\text{e-}03 y_{n-1} + 4.2619903069630594\,\text{e-}02 y_n\\
&\quad + 1.0451108327339836\,\text{e-}02 \Delta t f_n + 1.4505253390564679\,\text{e-}01 Y_2\\
&\quad + 3.7950229482122844\,\text{e-}02 \Delta t F_2 + 8.0490258839932849\,\text{e-}01 Y_7\\
&\quad + 2.1605179779405539\,\text{e-}01 \Delta t F_7
\end{aligned}
$$

HB37. Here $c(\text{HB37}) = 3.741$, $c_{\text{eff}}(\text{HB34}) = 0.534$, and
$\sigma = [0, 0.26730677262648872, 0.36663180051479927, 0.63393857314128788, 0.66822290349557856, 0.60553808135903842, 0.86781907493917998]^T$.

$$
Y_2 = 1.00\,\text{e+}00 y_n + 2.6730677262648872\,\text{e-}01 \Delta t f_n,
$$

$$
\begin{aligned}
Y_3 &= 1.3255018308632302\,\text{e-}01 y_{n-1} + 3.5431561651855202\,\text{e-}02 \Delta t f_{n-1} \\
&\quad + 8.6744981691367695\,\text{e-}01 Y_2 + 2.3187521097463354\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.00\,\text{e+}00 Y_3 + 2.6730677262648872\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 7.8339201835586181\,\text{e-}02 y_n + 2.0940599212805666\,\text{e-}02 \Delta t f_n \\
&\quad + 2.8923969641243608\,\text{e-}01 Y_2 \\
&\quad + 6.3242110175197763\,\text{e-}01 Y_4 + 1.6905044365020938\,\text{e-}01 \Delta t F_4, \\
Y_6 &= 4.5282440752517999\,\text{e-}01 y_n + 1.2104303094205779\,\text{e-}01 \Delta t f_n \\
&\quad + 4.1010199215761101\,\text{e-}02 Y_2 + 5.0616539325905896\,\text{e-}01 Y_5 \\
&\quad + 1.3530143768729652\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.4859000088371840\,\text{e-}02 Y_2 + 3.9719113580793941\,\text{e-}03 \Delta t F_2 \\
&\quad + 9.8514099991162807\,\text{e-}01 Y_6 + 2.6333486126840927\,\text{e-}01 \Delta t F_6, \\
y_{n+1} &= 8.3965945879398759\,\text{e-}04 y_{n-2} + 4.8613008172801175\,\text{e-}02 y_n \\
&\quad + 1.2874262041034118\,\text{e-}01 Y_2 + 3.4413774361365404\,\text{e-}02 \Delta t F_2 \\
&\quad + 8.2180471195806348\,\text{e-}01 Y_7 + 2.1967396528275113\,\text{e-}01 \Delta t F_7.
\end{aligned}
$$

HB47. Here $c(\mathrm{HB47}) = 3.749$, $c_{\mathrm{eff}}(\mathrm{HB34}) = 0.536$, and
$\sigma = [0, 0.26674074639327200, 0.36549307559105731, 0.63223382198432931, 0.66587063087515153, 0.60767834714088564, 0.86208353103336455]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 2.6674074639327200\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.3261467879184577\,\text{e-}01 y_{n-1} + 3.5373738403640963\,\text{e-}02 \Delta t f_{n-1} \\
&\quad + 8.6738532120815426\,\text{e-}01 Y_2 + 2.3136700798963106\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.00\,\text{e+}00 Y_3 + 2.6674074639327200\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 1.3020054056568347\,\text{e-}02 y_n + 3.4729789371297781\,\text{e-}03 \Delta t f_n \\
&\quad + 3.5567888199659903\,\text{e-}01 Y_2 \\
&\quad + 6.3130106394683261\,\text{e-}01 Y_4 + 1.6839371699604486\,\text{e-}01 \Delta t F_4, \\
Y_6 &= 4.6669675414872375\,\text{e-}01 y_n + 1.2448704054094795\,\text{e-}01 \Delta t f_n \\
&\quad + 2.1285468014294449\,\text{e-}02 Y_2 + 5.1201777783698676\,\text{e-}01 Y_5 \\
&\quad + 1.3657600422686239\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 3.6181290868897723\,\text{e-}02 Y_2 + 9.6510245318418703\,\text{e-}03 \Delta t F_2 \\
&\quad + 9.6381870913110224\,\text{e-}01 Y_6 + 2.5708972186143020\,\text{e-}01 \Delta t F_6, \\
y_{n+1} &= 2.1684068610740816\,\text{e-}04 y_{n-3} + 5.6279690153558326\,\text{e-}02 y_n \\
&\quad + 1.0817146260543896\,\text{e-}01 Y_2 + 2.8853736673826694\,\text{e-}02 \Delta t F_2 \\
&\quad + 8.3533200655489537\,\text{e-}01 Y_7 + 2.2281708291464233\,\text{e-}01 \Delta t F_7.
\end{aligned}
$$

HB28. Here $c(\mathrm{HB47}) = 4.424$, $c_{\mathrm{eff}}(\mathrm{HB34}) = 0.553$, and
$\sigma = [0, 0.22603296506803669, 0.45206593013607332, 0.57538108409035438, 0.40503033408031602, 0.53276080986754903, 0.75879377493558564, 0.92644778515340753]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 2.2603296506803669\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 2.2603296506803669\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 2.0286172312623845\,\text{e-}02 y_{n-1} + 1.0803874178805883\,\text{e-}01 y_n \\
&\quad + 4.5853436777034788\,\text{e-}03 \Delta t f_{n-1} \\
&\quad + 8.7167508589931730\,\text{e-}01 Y_3 + 1.9702730424175827\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 4.9460539841339118\,\text{e-}01 y_n \\
&\quad + 5.0539460158660887\,\text{e-}01 Y_4 + 1.1423584032600029\,\text{e-}01 \Delta t F_4, \\
Y_6 &= 2.3834939846072800\,\text{e-}01 y_n + 5.3874821256261370\,\text{e-}02 \Delta t f_n \\
&\quad + 4.3546190149383168\,\text{e-}03 Y_2 + 7.5729598252433372\,\text{e-}01 Y_5 \\
&\quad + 1.7117385636408730\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.0 Y_6 + 2.2603296506803666\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 3.3268281696016186\,\text{e-}02 Y_2 + 7.5197283544692212\,\text{e-}03 \Delta t F_2 \\
&\quad + 7.6309888786772917\,\text{e-}02 Y_3 \\
&\quad + 8.9042182951721116\,\text{e-}01 Y_7 + +2.0126468628708105\,\text{e-}01 \Delta t F_7, \\
y_{n+1} &= 3.0622000515870560\,\text{e-}03 y_{n-1} + 3.4396037347807322\,\text{e-}02 y_n \\
&\quad + 2.2397284277651699\,\text{e-}01 Y_3 + 5.0625245747493351\,\text{e-}02 \Delta t F_3 \\
&\quad + 7.3856891982408845\,\text{e-}01 Y_8 + 1.6694092285493578\,\text{e-}01 \Delta t F_8.
\end{aligned}
$$

HB38. Here $c(\mathrm{HB38}) = 4.431$, $c_{\mathrm{eff}}(\mathrm{HB38}) = 0.5538$, and
$\sigma = [0, 0.22569317768029287, 0.45138635536058574, 0.55305470898706099, 0.41140419740313083, 0.53185945085898290, 0.75755262853927563, 0.91830243658955979]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 2.2569317768029287\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 2.2569317768029287\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.8136891011337745\,\text{e-}02 y_{n-1} + 1.4429794897581311\,\text{e-}01 y_n \\
&\quad + 4.0933725655899564\,\text{e-}03 \Delta t f_{n-1} \\
&\quad + 8.3756516001284920\,\text{e-}01 Y_3 + 1.8903274247760293\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 4.7171067241834541\,\text{e-}01 y_n \\
&\quad + 5.2828932758165470\,\text{e-}01 Y_4 + 1.1923129707648883\,\text{e-}01 \Delta t F_4, \\
Y_6 &= 2.2802085154938864\,\text{e-}01 y_n + 5.1462750563548147\,\text{e-}02 \Delta t f_n \\
&\quad + 2.7780924146448754\,\text{e-}02 Y_2 + 7.4419822430416260\,\text{e-}01 Y_5 \\
&\quad + 1.6796046206723791\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.0 Y_6 + 2.2569317768029284\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 7.2741873790442308\,\text{e-}04 Y_2 + 1.6417344646181553\,\text{e-}04 \Delta t F_2 \\
&\quad + 1.2137884359347230\,\text{e-}01 Y_3 \\
&\quad + 8.7789373766862333\,\text{e-}01 Y_7 + +1.9813462732006101\,\text{e-}01 \Delta t F_7, \\
y_{n+1} &= 4.3049458217011075\,\text{e-}04 y_{n-2} + 4.4222997383088054\,\text{e-}02 y_n \\
&\quad + 1.9714726015247588\,\text{e-}01 Y_3 + 4.4494791614775667\,\text{e-}02 \Delta t F_3 \\
&\quad + 7.5819924788226600\,\text{e-}01 Y_8 + 1.7112039756935671\,\text{e-}01 \Delta t F_8.
\end{aligned}
$$

HB48. Here $c(\mathrm{HB48}) = 4.436$, $c_{\mathrm{eff}}(\mathrm{HB48}) = 0.5544$, and
$\sigma = [0, 0.22544870448684559, 0.45089740897369118, 0.54156810616463591, 0.41374101214197817, 0.53175265896477997, 0.75720136345162536, 0.91365101994262776]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 2.2544870448684559\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 2.2544870448684559\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.7045241242834696\,\text{e-}02 y_{n-1} + 1.6270827724326312\,\text{e-}01 y_n \\
&\quad + 3.8428275558628319\,\text{e-}03 \Delta t f_{n-1} \\
&\quad + 8.2024648151390223\,\text{e-}01 Y_3 + 1.8492350661720261\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 4.6058416660964840\,\text{e-}01 y_n \\
&\quad + 5.3941583339035160\,\text{e-}01 Y_4 + 1.2161060081754689\,\text{e-}01 \Delta t F_4, \\
Y_6 &= 1.9450747482843252\,\text{e-}02 y_n + 4.3851458213079730\,\text{e-}03 \Delta t f_n \\
&\quad + 2.4022149798258005\,\text{e-}01 Y_2 + 7.4032775453457667\,\text{e-}01 Y_5 \\
&\quad + 1.6690593315547575\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.0 Y_6 + 2.2544870448684556\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 1.2975778650579978\,\text{e-}01 Y_2 + 2.9253724864813244\,\text{e-}02 \Delta t F_2 \\
&\quad + 8.7024221349420028\,\text{e-}01 Y_7 + +1.9619497962203228\,\text{e-}01 \Delta t F_7, \\
y_{n+1} &= 1.1810143414334793\,\text{e-}04 y_{n-3} + 4.6906182658833978\,\text{e-}02 y_n \\
&\quad + 1.8407219102382708\,\text{e-}01 Y_3 + 4.1498836998376983\,\text{e-}02 \Delta t F_3 \\
&\quad + 7.6890352488319558\,\text{e-}01 Y_8 + 1.7334830356028547\,\text{e-}01 \Delta t F_8.
\end{aligned}
$$

HB29. Here $c(\mathrm{HB29}) = 5.271$, $c_{\mathrm{eff}}(\mathrm{HB29}) = 0.586$, and
$\sigma = [0, 0.18973510416279940, 0.37947020832559880, 0.52900811739988363, 0.33177283195412421, 0.52150793611692348, 0.63723279555722223, 0.75047127700234584, 0.94020638116514521]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 1.8973510416279943\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 1.8973510416279937\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 7.0619852286311971\,\text{e-}02 y_n \\
&\quad + 9.2938014771368804\,\text{e-}01 Y_3 + 1.7633603913329446\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 5.3839866310977136\,\text{e-}01 y_n \\
&\quad + 4.6160133689022864\,\text{e-}01 Y_4 + 8.7581977736554989\,\text{e-}02 \Delta t F_4, \\
Y_6 &= 1.0 Y_5 + 1.8973510416279937\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.4191585515183294\,\text{e-}01 y_n + 2.6926419559585760\,\text{e-}02 \Delta t f_n, \\
&\quad + 8.5808414484816709\,\text{e-}01 Y_6 + 1.6280868460321360\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 1.7093368412148527\,\text{e-}01 Y_2 + 3.2432120361721037\,\text{e-}02 \Delta t F_2 \\
&\quad + 9.3714265975099066\,\text{e-}06 Y_3 \\
&\quad + 8.2905694445191724\,\text{e-}01 Y_7 + +1.5730120571247669\,\text{e-}01 \Delta t F_7, \\
Y_9 &= 1.0 Y_8 + 1.8973510416279937\,\text{e-}01 \Delta t F_8, \\
y_{n+1} &= 3.7795761833689869\,\text{e-}03 y_{n-1} + 1.2870630630322917\,\text{e-}03 y_n \\
&\quad + 2.0055779156521011\,\text{e-}01 Y_3 + 3.8052853473286137\,\text{e-}02 \Delta t F_3 \\
&\quad + 1.3274175171758950\,\text{e-}02 Y_4 \\
&\quad + 7.8110139401572964\,\text{e-}01 Y_9 + 1.4820235435528228\,\text{e-}01 \Delta t F_9.
\end{aligned}
$$

HB39. Here $c(\mathrm{HB39}) = 5.279$, $c_{\mathrm{eff}}(\mathrm{HB39}) = 0.587$, and
$\sigma = [0, 0.18943721061280203, 0.37887442122560405, 0.51190473654655466, 0.33220975334717834, 0.52164696395998034, 0.64258973999314262, 0.74627464684283940, 0.93064568637963851]^T$.

$$
\begin{aligned}
Y_2 &= 1.00\,\text{e+}00 y_n + 1.8943721061280203\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 1.8943721061280203\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 9.9253459073824102\,\text{e-}02 y_n \\
&\quad + 9.0074654092617590\,\text{e-}01 Y_3 + 1.7063491218218491\,\text{e-}01 \Delta t F_3,
\end{aligned}
$$

$$
\begin{aligned}
Y_5 &= 5.2632270935350933\,\text{e-}01 y_n \\
&\quad + 4.7367729064649067\,\text{e-}01 Y_4 + 8.9732104670700696\,\text{e-}02 \Delta t F_4, \\
Y_6 &= 1.0 Y_5 + 1.8943721061280203\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.3130188311917806\,\text{e-}01 y_n + 2.4873462486305245\,\text{e-}02 \Delta t f_n, \\
&\quad + 2.3116942601563637\,\text{e-}06 Y_2 \\
&\quad + 8.6869580518656186\,\text{e-}01 Y_6 + 1.6456331020558435\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 1.8332627761890183\,\text{e-}01 Y_2 + 3.4728818664152943\,\text{e-}02 \Delta t F_2 \\
&\quad + 5.9086889835505332\,\text{e-}03 Y_3 \\
&\quad + 8.1076503339754757\,\text{e-}01 Y_7 + +1.5358906638922670\,\text{e-}01 \Delta t F_7, \\
Y_9 &= 1.3789243236016317\,\text{e-}02 Y_3 + 2.6121957750924324\,\text{e-}03 \Delta t F_3, \\
&\quad + 9.8621075676398373\,\text{e-}01 Y_8 + 1.8682501483770966\,\text{e-}01 \Delta t F_8, \\
y_{n+1} &= 4.9675940773150854\,\text{e-}04 y_{n-2} + 1.7898560493677686\,\text{e-}02 y_n \\
&\quad + 1.7826598991396123\,\text{e-}01 Y_3 + 3.3770211876430631\,\text{e-}02 \Delta t F_3 \\
&\quad + 2.0231601518094915\,\text{e-}04 Y_4 \\
&\quad + 8.0313637416944861\,\text{e-}01 Y_9 + 1.5214391446434000\,\text{e-}01 \Delta t F_9.
\end{aligned}
$$

HB210. Here $c(\text{HB210}) = 6.102$, $c_{\text{eff}}(\text{HB210}) = 0.610$, and
$\sigma = [0, 0.16199536174015935, 0.32587309121796598, 0.48975082069577258, 0.39110537738046303, 0.55498310685826957, 0.47054816607752142, 0.63442589555532791, 0.79830362503313468, 0.96218135451094122]^T$.

$$
\begin{aligned}
Y_2 &= 9.9838267569696360\,\text{e-}01 y_n + 1.6361268604319576\,\text{e-}01 \Delta t f_n, \\
&\quad + 1.6173243030364048\,\text{e-}03 y_{n-1}, \\
Y_3 &= 1.0 Y_2 + 1.6387772947780663\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.0 Y_3 + 1.6387772947780663\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 4.0163969692480539\,\text{e-}01 y_n \\
&\quad + 5.9836030307519472\,\text{e-}01 Y_4 + 9.8057927877615150\,\text{e-}02 \Delta t F_4, \\
Y_6 &= 1.0 Y_5 + 1.6387772947780663\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 4.3651141473686855\,\text{e-}01 y_n + 7.1534499538223190\,\text{e-}02 \Delta t f_n, \\
&\quad + 1.2142963792919311\,\text{e-}02 Y_2 + 1.9899613355148157\,\text{e-}03 \Delta t F_2, \\
&\quad + 3.2677181024377802\,\text{e-}03 Y_3 \\
&\quad + 5.4807790336777440\,\text{e-}01 Y_6 + 8.9817762380867569\,\text{e-}02 \Delta t F_6, \\
Y_8 &= 1.0 Y_7 + +1.6387772947780663\,\text{e-}01 \Delta t F_7, \\
Y_9 &= 1.0 Y_8 + 1.6387772947780663\,\text{e-}01 \Delta t F_8, \\
Y_{10} &= 1.0 Y_9 + 1.6387772947780663\,\text{e-}01 \Delta t F_9, \\
y_{n+1} &= 2.7123097803231889\,\text{e-}03 y_{n-1} \\
&\quad + 6.9146647649389378\,\text{e-}03 Y_3 + 1.1331595617783755\,\text{e-}03 \Delta t F_3 \\
&\quad + 2.1863321246897169\,\text{e-}01 Y_4 + 2.3225424910426631\,\text{e-}02 \Delta t F_4 \\
&\quad + 7.7173981298576633\,\text{e-}01 Y_{10} + 1.2647096829973448\,\text{e-}01 \Delta t F_{10}.
\end{aligned}
$$

HB310. Here $c(\text{HB310}) = 6.142$, $c_{\text{eff}}(\text{HB310}) = 0.614$, and
$\sigma = [0, 0.16282659661384036, 0.32565319322768072, 0.48847978984152113, 0.35907995652089209, 0.48552698907685793, 0.55293076340580660, 0.62414211119929675, 0.78175633090122820, 0.94458292751506845]^T$.

$$
\begin{aligned}
Y_2 &= 1.0 y_n + 1.6282659661384036\,\text{e-}01 \Delta t f_n, \\
Y_3 &= 1.0 Y_2 + 1.6282659661384036\,\text{e-}01 \Delta t F_2, \\
Y_4 &= 1.0 Y_3 + 1.6282659661384036\,\text{e-}01 \Delta t F_3, \\
Y_5 &= 4.4867735985957147\,\text{e-}01 y_n \\
&\quad + 5.5132264014042853\,\text{e-}01 Y_4 + 8.9769989130223021\,\text{e-}02 \Delta t F_4, \\
Y_6 &= 6.9705129853931833\,\text{e-}02 y_n \\
&\quad + 9.3029487014606804\,\text{e-}01 Y_5 + 1.5147674755319884\,\text{e-}01 \Delta t F_5, \\
Y_7 &= 1.9509480297883888\,\text{e-}01 y_n + 3.1766622786092102\,\text{e-}02 \Delta t f_n, \\
&\quad + 1.4397346095268116\,\text{e-}03 Y_2, \\
&\quad + 8.0346546241163441\,\text{e-}01 Y_6 + 1.3082554674125191\,\text{e-}01 \Delta t F_6, \\
Y_8 &= 3.3918333816126425\,\text{e-}02 Y_2 + 5.5228068580921580\,\text{e-}03 \Delta t F_2, \\
&\quad + 2.0092983397903494\,\text{e-}01 Y_3, \\
&\quad + 7.6515183220483873\,\text{e-}01 Y_7 + +1.2458706873075814\,\text{e-}01 \Delta t F_7, \\
Y_9 &= 1.4299235887157244\,\text{e-}02 Y_3 + 2.3282959136844139\,\text{e-}03 \Delta t F_3, \\
&\quad + 3.1633116249260595\,\text{e-}03 Y_4, \\
&\quad + 9.8253745248791668\,\text{e-}01 Y_8 + +1.5998322943424034\,\text{e-}01 \Delta t F_8, \\
Y_{10} &= 1.0 Y_9 + 1.6282659661384036\,\text{e-}01 \Delta t F_9, \\
y_{n+1} &= 4.9225322847329942\,\text{e-}04 y_{n-2} + 1.7871006145269043\,\text{e-}04 y_n \\
&\quad + 2.4021344291524174\,\text{e-}03 Y_3 + 3.9113137370776901\,\text{e-}04 \Delta t F_3 \\
&\quad + 1.6901708746032448\,\text{e-}01 Y_4 + 4.1446692144727487\,\text{e-}04 \Delta t F_4 \\
&\quad + 8.2790981482059733\,\text{e-}01 Y_{10} + 1.3480573745043264\,\text{e-}01 \Delta t F_{10}.
\end{aligned}
$$

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Huong Nguen-Thu, Truong Nguen-Ba, Tjeri Giordanos, Remi Vajenkurs. Ermita-Birkhofa k-soļu 4. kārtas laika diskretizācija no 4. līdz 10. etapam, kas saglabā stingru stabilitāti
Nestacionāru parciālo vienādojumu diskretizācija pēc telpiskās koordinātas ar taišņu metodi noved pie parastu diferenciālvienādojumu sistēmas. Rakstā aplūkota daudzsoļu metožu saime iegūtās parasto diferenciālvienādojumu sistēmas skaitliskai integrēšanai. Nelineāras stabilitātes īpašība, kas piemīt laika diskretizācijas metodēm, kuri saglabā stingro stabilitāti (SSS), ir īpaši piemērota hiperbolisko nezūdamības likumu integrēšanai. Rakstā ir konstruēta k soļu atklāto Ermita-Birkhoffa metožu saime, kuri saglabā stingro stabilitāti, ar kārtu 4 ar nenegatīviem koeficientiem. Metodes ir konstruētas uz lineāras k soļu ceturtās kārtas metodes kombinēšanu ar ceturtās kārtas no ceturtā līdz desmitam posmam Runges-Kutta metodēm. Metodes ir precīzi formulētas un saglabā stingru stabilitāti. Šīs jaunās metodes saglabā monotonitāti un aizkavē kļūdas pieaugšanu, tādējādi tās ir piemērotas parciālo diferenciālvienādojumu risināšanai ar taišņu metodi.

Piedāvātām metodēm ir lielākā efektivitāte attiecībā pret Kuranta-Fridriksa-Levi (KFL) koeficientiem. Visām jaunajām Ermita-Birkhofa metodēm ir lielāks efektīvais SSS koeficients un lielāki maksimālās efektivitātes KFL skaitļi, salīdzinot ar tās pašas kārtas Huanga hibrīda metodēm un Runges-Kutta metodēm, ja tās pielieto neviskoziem Burgera vienādojumiem. Rakstā aplūkoti divi piemēri, kuros Birgera vienādojums risināts skaitliski ar piedāvātajiem metodēm.

Хуонг Нгуен-Ху, Труонг Нгуен-Ба, Тьери Джордано, Реми Вайенкур. К-шаговые, от 4 до 10 этапов, дискретизации по времени Эрмита-Биркхофа 4 порядка, сохраняющие сильную устойчивость
Дискретизация нестационарных уравнений в частных производных по пространственной координате с помощью метода прямых приводит к системе обыкновенных дифференциальных уравнений. В статье рассматривается семейство многошаговых методов для численного интегрирования полученной системы обыкновенных дифференциальных уравнений. Нелинейное свойство устойчивости методов дискретизации по времени, сохраняющих сильную устойчивость, делает их особенно привлекательными для интегрирования гиперболических законов сохранения. В статье приводится набор к-шаговых явных, сохраняющих сильную устойчивость (ССУ), методов Эрмита-Биркхофа (ССУ ЭБ) четвертого порядка с неотрицательными коэффициентами, построенных по принципу комбинирования к-шаговых методов четвертого порядка с четырех – десятиэтапными методами Рунге-Кутта четвертого порядка. Метод точно сформулирован и сохраняет сильную устойчивость. Предлагаемые методы сохраняют монотонность и задерживают возрастание ошибки, поэтому они применимы для решения дифференциальных уравнений в частных производных методом прямых. Вообще говоря, у новых методов Эрмита-Биркхофа более эффективные коэффициенты Куранта-Фридрикса-Леви (КФЛ). У всех новых методов Эрмита-Биркхофа эффективный коэффициент ССУ и числа максимальной эффективности КФЛ больше, чем у гибридного метода Хуанга того же порядка, так же они больше по сравнению с методами Рунге-Кутта, если их применять для решения невязких уравнений Бюргера. В статье рассмотрены два примера, в которых численно решается с помощью предложенных методов уравнение Бюргера.