Strong-stability-preserving, One-step, 9-stage, Hermite–Birkhoff–Taylor, Time-discretization Methods Combining Taylor and RK4 Methods

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***Abstract* – The ODE solver HBT(12)4 of order 12 (Can. Appl. Math. Q. 16(1) (2008) 77–94), which combines a Taylor series method of order 9 with a Runge–Kutta method of order 4, is expanded into optimal, one-step, 9-stage, explicit, strongstability-preserving (SSP), Hermite–Birkhoff–Taylor methods, HBT(*p*), of orders *p* = 6, 7, . . . , 12, with nonnegative coefficients. These methods are constructed by combining Taylor methods, T(*p* − 3), of orders *p* − 3 with a 9-stage Runge–Kutta method, RK(9, 4), of order 4. Several new one-step SSP methods arise with higher order than those appearing in the recent literature. The Shu–Osher form of RK methods is extended to the above combined methods. Compared to Huang's *k*-step hybrid methods, HM(*k*, *p*), of the same order, the new HBT(*p*) generally have larger effective SSP coefficients and larger maximum effective CFL numbers on Burgers' equation, independently of the number *k* of steps of HM(*k*, *p*). The new HBT(*p*) are listed in their canonical Shu–Osher form in the appendix.**

***Keywords* – Strong stability preserving; Hermite–Birkhoff-Taylor method; SSP coefficient; time discretization; method of lines; comparison with other SSP methods.**

I. INTRODUCTION

We are concerned with the numerical solution of initial value problems

$$\frac{dy}{dt} = f(t, y(t)), \qquad y(t_0) = y_0. \tag{1}$$

where the function $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is such that

$$\|y(t + \Delta t)\| \le \|y(t)\|, \tag{2}$$

for all $\Delta t \ge 0$. Here $\|\cdot\|$ may be a norm or, more generally, any convex functional.

It is assumed that *f* of (1) satisfies the discrete analog of (2),

$$\|y_n + \Delta t f(t_n, y_n)\| \le \|y_n\|, \tag{3}$$

for the forward Euler method (FE) with $\Delta t \le \Delta t_{\text{FE}}$ where the vector y_n is a numerical approximation to $y(t_0 + n\Delta t)$.

Recently, new strong-stability-preserving Runge–Kutta (SSP RK) methods have been developed for solving (1) such that

$$\|y_{n+1}\| \le \|y_n\|, \tag{4}$$

for $0 \le \Delta t \le \Delta t_{\max} = c(\text{SSP RK})\Delta t_{\text{FE}}$ whenever inequality (3) holds. Here the number *c*(SSP RK), called the SSP coefficient of SSP RK, depends only on the numerical integration method but not on the vector function *f*.

To consider the Hermite–Birkhoff–Taylor methods of order *p*, (HBT(*p*)), FE is expanded into a series method of order at least 1, denoted by S(*p* − 3),

$$\begin{aligned} y_{n+1} &= y_n + \sum_{m=1}^{p-3} \eta_m (\Delta t)^m f^{(m-1)}(t_n, y_n) \\ &= y_n + (\eta_1 \Delta t) f(t_n, y_n) \\ &\quad + \sum_{m=2}^{p-3} \frac{\eta_m}{\eta_1^m} (\eta_1 \Delta t)^m f^{(m-1)}(t_n, y_n) \end{aligned} \tag{5}$$

of degree *p* − 3 in Δt where $\eta_m \le \frac{1}{m!}$. It is to be noted that S(*p* − 3) is the usual Taylor method, T(*p* − 3), if the coefficients $\eta_m = \frac{1}{m!}$, $m = 1, 2, \dots, p-3$. Hence, in general, S(*p* − 3) is similar, but not identical, to the usual Taylor method, T(*r*), of order *r*, *r* = 1, 2, 3, . . . , *p* − 3, if S(*p* − 3) has an error of order 2, 3, . . . , *p* − 2, respectively.

Moreover, it is assumed that, besides the discrete analog (3), *f* of (1) also satisfies

$$\begin{aligned} &\left\| y_n + \sum_{m=1}^{p-3} \eta_m (\Delta t)^m f^{(m-1)}(t_n, y_n) \right\| \\ &\qquad \le \left\| y_n + \sum_{m=1}^{p-3} \frac{(\Delta t)^m}{m!} f^{(m-1)}(t_n, y_n) \right\| \le \|y_n\|, \end{aligned} \tag{6}$$

for method S(*p* − 3) of (5) with $\Delta t \le \Delta t_{\text{S}(p-3)}$.

The number

$$\Delta t_{\text{FE},\text{S}(p-3)} = \min\{\Delta t_{\text{FE}}, \Delta t_{\text{S}(p-3)}\} \tag{7}$$

stands for the maximal time step for which inequalities (3) and (6) hold.

Similar to the expansion of usual RKs into SSP RKs, the main contribution of this paper is an expansion of the HBT ODE solver HBT(12)4, developed in [14], into SSP HBT methods of order *p* (SSP HBT(*p*)) that preserve the absolute monotonicity property (4) for $0 \le \Delta t \le \Delta t_{\max} = c(\text{SSP HBT}(p))\Delta t_{\text{FE},S(p-3)}$ whenever inequalities (3) and (6) hold. Here the number *c*(SSP HBT(*p*)) is called the SSP coefficient of our SSP HBT(*p*) methods.

The monotonicity property (4) is suitable to avoid error growth as it follows property (2) of the true solution.

The main application of such monotonicity results are found in the numerical solution of hyperbolic PDEs, in particular, of conservation laws. For the one-dimensional equation

$$y_t + g(y)_x = 0, \quad y(x,0) = y_0(x), \tag{8}$$

the spatial derivative $g(y)_x$ can be approximated by a conservative finite difference or finite element at $x_j, j = 1, 2, \dots, N$, (see, for example, [7], [15], [21], [1]). Such spatial semidiscretization will lead to the ODE system (1).

In this paper, to solve system (1), we construct new explicit, SSP, one-step, multiderivative, 9-stage, general linear methods of order *p*, *p* = 6, 7, . . . , 12, with nonnegative coefficients as a combination of Taylor methods of order *p*− 3 and a 9-stage RK method of order 4 (RK(9,4)).

We shall denote our new SSP 9-stage methods of order *p* as HBT(*p*) or, more explicitly, HBT(9, *p*) since HB interpolation polynomials enter in their construction as it is briefly sketched in Section II (see [14] for fuller developments). The objective of such high-order methods is to maintain the absolute monotonicity property (4) while achieving higher-order accuracy in time, perhaps with a modified time-step restriction, measured here with the SSP coefficient *c*(HBT(*p*)):

$$\Delta t \le c(\text{HBT}(p))\Delta t_{\text{FE},S(p-3)}. \tag{9}$$

The SSP coefficient describes the ratio of the maximal HBT(*p*) time step to the time step $\Delta t_{\text{FE},S(p-3)}$, for which the two conditions (3) and (6) hold.

A brief review of the development of SSP methods will appear in Section V on the construction of HBT(*p*).

The new HBT(*p*) have larger effective SSP coefficients than known *k*-step SSP hybrid methods (HM(*k*, *p*)) of the same order *p*. In particular, no counterparts of one-step HBT(*p*) methods of order greater than 8 have been found in the literature among hybrid and general, linear multistep, multistage methods.

Section II introduces 9-stage HBT(*p*) methods. Order conditions are listed in Section III. Section IV derives the Shu– Osher form of HBT(*p*). In Section V, several new HBT(*p*) methods are constructed by a computer search. Section VI presents numerical results for several SSP methods applied to Burgers' equations. Seven of the new HBT(*p*) methods are listed in the Appendix in their Shu–Osher form.

II. NINE-STAGE HBT(*P*) METHODS

Nine-stage HBT(*p*) methods are constructed, as a subclass of general linear methods, by the following ten formulae to perform integration from t_n to t_{n+1}.

Let Δt denote the step size. The abscissa vector $[c_1, c_2, c_3, \dots, c_9]^T$ defines the eight off-step points $t_n + c_j\Delta t$. In all cases, $c_1 = 0$ and, by convention, $c_1^0 = 1$.

Let $F_1 = f_n$ and $F_j := f(t_n + c_j\Delta t, Y_j)$, $j = 2, 3, \dots, 9$, denote the *j*th stage derivatives.

With the initial stage value, $Y_1 = y_n$, HB polynomials are used as predictors P_i to obtain the stage values Y_i to order *p* − 3,

$$Y_i = y_n + \Delta t \sum_{j=1}^{i-1} a_{i,j}F_j + \sum_{m=2}^{p-3} (\Delta t)^m \gamma_{i,m}\, y_n^{(m)},$$
$$i = 2, 3, \dots, 9.. \tag{10}$$

An HB polynomial is used as integration formula to obtain y_{n+1} to order *p*,

$$y_{n+1} = y_n + \Delta t \sum_{j=1}^{9} b_jF_j + \sum_{m=2}^{p-3} (\Delta t)^m \gamma_{10,m}\, y_n^{(m)}. \tag{11}$$

Formulae (10)–(11) are the Butcher form of HBT(*p*).

One sees that the derivatives $y_n^{(m)}$, $m = 2, 3, \dots, p-3$, are computed only once per step at $t = t_n$. The defining formulae of HBT(*p*) involve the usual RK parameters c_i, $a_{i,j}$ and b_j and the Taylor expansion parameters $\gamma_{i,j}$. Then, we can represent an HBT(*p*) method by its *coefficient scheme* (A, b, γ_0) where *A* denotes the 9×9 matrix $A = (a_{i,j})$, b the 9-vector $b = (b_1, b_2, \dots, b_9)^T$ and γ_0 the $10 \times (p-4)$ matrix $\gamma_0 = (\gamma_{i,j})$ of Taylor expansion parameters $\gamma_{i,j}$. One can display the coefficient scheme (A, b, γ_0) and the c_i in the Butcher tableau

$$\begin{array}{c|ccccc}
c_1 & & & & & \\
c_2 & a_{2,1} & & & & \\
c_3 & a_{3,1} & a_{3,2} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
c_9 & a_{9,1} & a_{9,2} & \cdots & a_{9,8} & \\
\hline
 & b_1 & b_2 & \cdots & b_8 & b_9
\end{array}$$

and the $10 \times (p-4)$ matrix γ_0.

$$\gamma_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \gamma_{2,2} & \gamma_{2,3} & \cdots & \gamma_{2,p-3} \\ \gamma_{3,2} & \gamma_{3,3} & \cdots & \gamma_{3,p-3} \\ \vdots & & \cdots & \vdots \\ \gamma_{10,2} & \gamma_{10,3} & \cdots & \gamma_{10,p-3} \end{bmatrix}. \quad (12)$$

Notation 1: We shall denote the SSP methods used in this paper as follows:

- HBT(p) or HBT(9, p) for one-step, 9-stage, HBT methods of order p.
- HM(k, p) for k-step hybrid methods of order p.
- LM(k, p) for linear k-step methods of order p.
- RK(s, p) for s-stage RK methods of order p.
- TSRK(s, p) for 2-step, s-stage RK methods of order p.

In a table head, OM(k, p) designates any of the other (k, p) methods, except HBT(p), appearing under it.

All the methods considered in this work are SSP. Therefore the denomination SSP will often be omitted.

III. ORDER CONDITIONS FOR HBT(9, P)

We impose the following simplifying assumptions on HBT(p) (see [14]):

$$\sum_{j=1}^{i-1} a_{i,j} c_j^k + k!\gamma_{i,k+1} = \frac{1}{k+1} c_i^{k+1}, \quad (13)$$

for $i = 2, 3, \ldots, 9$ and $k = 0, 1, \ldots, p - 4$.

There remain six sets of equations to be solved:

$$\sum_{i=1}^{9} b_i c_i^k + k!\gamma_{10,k+1} = \frac{1}{k+1}, \quad k = 0, 1, \ldots, p-4, \quad (14)$$

$$\sum_{i=1}^{9} b_i c_i^k = \frac{1}{k+1}, \quad k = p-3, p-2, p-1, \quad (15)$$

$$\sum_{i=2}^{9} b_i \left[\sum_{j=1}^{i-1} a_{i,j} \frac{c_j^{p-3}}{(p-3)!}\right] = \frac{1}{(p-1)!}, \quad (16)$$

$$\sum_{i=2}^{9} b_i \frac{c_i}{p-1} \left[\sum_{j=1}^{i-1} a_{i,j} \frac{c_j^{p-3}}{(p-3)!}\right] = \frac{1}{p!}, \quad (17)$$

$$\sum_{i=2}^{9} b_i \left[\sum_{j=1}^{i-1} a_{i,j} \frac{c_j^{p-2}}{(p-2)!}\right] = \frac{1}{p!}, \quad (18)$$

$$\sum_{i=2}^{9} b_i \left[\sum_{j=1}^{i-1} a_{i,j} \left[\sum_{k=1}^{j-1} a_{j,k} \frac{c_k^{p-3}}{(p-3)!}\right]\right] = \frac{1}{p!}. \quad (19)$$

IV. HBT(9, P) IN MODIFIED SHU–OSHER FORM

We have the following modified Shu–Osher form of HBT(p) which is a generalization of the Shu–Osher form for RK methods (see [17], [4]),

$$\begin{aligned} Y_i &= v_i y_n + \left[\sum_{j=1}^{i-1} \alpha_{i,j} Y_j + \Delta t\, \beta_{i,j} F_j\right] \\ &\quad + \sum_{m=2}^{p-3} (\Delta t)^m \delta_{i,m}\, y_n^{(m)}, \quad i = 2, 3, \ldots, 10, \end{aligned} \quad (20)$$

$$y_{n+1} = Y_{10}.$$

We can rearrange (20) as follows:

$$\begin{aligned} Y_i &= v_i y_n + \sum_{j=2}^{i-1} \alpha_{i,j} \left[Y_j + \Delta t \frac{\beta_{i,j}}{\alpha_{i,j}} F_j\right] \\ &\quad + \alpha_{i,1} \left[y_n + \Delta t \frac{\beta_{i,1}}{\alpha_{i,1}} f_n \right. \\ &\quad \left. + \sum_{m=2}^{p-3} (\Delta t)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, y_n^{(m)}\right], \quad i = 2, 3, \ldots, 10, \end{aligned} \quad (21)$$

$$y_{n+1} = Y_{10}.$$

Here, consistency requires that

$$v_i + \sum_{j=1}^{i-1} \alpha_{i,j} = 1, \qquad i = 2, 3, \ldots, 10. \quad (22)$$

If all the coefficients $v_i, \alpha_{i,j}, \beta_{i,j}, \delta_{i,m}$ are nonnegative and suitable conditions are imposed on $\delta_{i,m}$, $i = 2, 3, \ldots, 10$ and $m = 2, 3, \ldots, p - 3$, the norm $\|Y_i\|$ of (21) can be expressed as less than or equal to convex combinations of norms of the FE and S(p − 3) methods, with a modified time step. The following straightforward extension of a result presented in [5], [8] holds.

Theorem 1: If f satisfies conditions (3) of the forward Euler and (6) of the series S(p − 3) methods, then the one - step, 9-stage HBT(p) methods (20) satisfy the monotonicity property

$$\|y_{n+1}\| \leq y_n\|$$

provided

$$\Delta t \leq c_{\text{feasible}} \Delta t_{\text{FE},S(p-3)},$$

where

- c_{feasible} is the minimum:
$$\min_{j=1,2,\ldots,i-1} \left\{\frac{\alpha_{i,j}}{\beta_{i,j}}\right\}, \quad i = 2, 3, \ldots, 10, \quad (23)$$
- the following conditions are imposed on $\delta_{i,m}$, $i = 2, 3, \ldots, 10$ and $m = 2, 3, \ldots, p - 3$:
$$\frac{\delta_{i,m}}{\alpha_{i,1}} \leq \left[\frac{1}{c_{\text{feasible}}}\right]^m \frac{1}{m!}, \quad (24)$$

with the convention that $a/0 = +\infty$, under the assumption that all coefficients of (20) are nonnegative.

Proof: Each stage of the HBT(p) method (21) can be rewritten as a convex combination of forward Euler steps and $y_n + \Delta t \frac{\beta_{i,1}}{\alpha_{i,1}} f_n + \sum_{m=2}^{p-3} (\Delta t)^m \frac{\delta_{i,m}}{\alpha_{i,1}}\, y_n^{(m)}$. We have by the convexity of $\|\cdot\|$ that

$$
\begin{aligned}
\|Y_i\| &= \left\| v_i y_n + \sum_{j=2}^{i-1} \alpha_{i,j} \left[Y_j + \Delta t \frac{\beta_{i,j}}{\alpha_{i,j}} F_j \right] \right. \\
&\quad \left. + \alpha_{i,1} \left[y_n + \Delta t \frac{\beta_{i,1}}{\alpha_{i,1}} f_n + \sum_{m=2}^{p-3} (\Delta t)^m \frac{\delta_{i,m}}{\alpha_{i,1}} y_n^{(m)} \right] \right\| \\
&\le v_i \|y_n\| + \sum_{j=2}^{i-1} \alpha_{i,j} \left\| Y_j + \frac{\Delta t}{c_{\text{feasible}}} F_j \right\| \\
&\quad + \alpha_{i,1} \left\| y_n + \frac{\Delta t}{c_{\text{feasible}}} f_n + \sum_{m=2}^{p-3} \left[\frac{\Delta t}{c_{\text{feasible}}} \right]^m \frac{1}{m!} y_n^{(m)} \right\| \\
&\le v_i \|y_n\| + \sum_{j=2}^{i-1} \alpha_{i,j} \|y_n\| + \alpha_{i,1} \|y_n\| .
\end{aligned}
$$

Thus, by (22), we obtain $\|Y_i\| \le \|y_n\|$ for $i = 2, 3, \ldots, 10$. In particular, this yields $\|y_{n+1}\| \le \|y_n\|$ for $i = 10$.

It is to be noted that each representation, v_i, $\alpha_{i,j}$, $\beta_{i,j}$ and $\delta_{i,m}$, of HBT(p) (21), which satisfies the conditions of Theorem 1, will produce a so-called feasible SSP coefficient, c_{feasible}, defined in this theorem. What we really want is not a feasible HBT(p) with a feasible SSP coefficient, c_{feasible}, but one with the largest SSP coefficient. This question will be considered in subsection IV-C.

Transforming formulae (10)–(11) into the Shu–Osher form of HBT(p) (20) and vice versa will be considered in subsection IV-B.

The next three subsections, IV-A, IV-B and IV-C, describe the generalized result for the new HBT(p), following closely the result for RK methods found in [4, Sections 3.1 to 3.4].

A. HBT(9, p) in compact vector notation

In the following sections, it will be helpful to represent an HBT(p) method in a more compact Shu–Osher form. To this end, we define vectors $\boldsymbol{v} \in \mathbb{R}^{10}$,

$$\boldsymbol{v} = [0, v_2, v_3, \ldots, v_{10}]^T,$$

strictly lower triangular matrices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{10\times 10}$, and rectangular matrices $\boldsymbol{\delta} \in \mathbb{R}^{10\times(p-4)}$ with zero first row, with components $v_i, \alpha_{i,j}, \beta_{i,j}, \delta_{i,j}$ coming from equation (20). Moreover, $\boldsymbol{Y}, \boldsymbol{F} \in \mathbb{R}^{10\times N}$ and $\boldsymbol{f}_{\mathrm{B}} \in \mathbb{R}^{(p-4)\times N}$:

$$
\begin{gathered}
\boldsymbol{Y} = [Y_1, Y_2, \ldots, Y_{10}]^T, \qquad \boldsymbol{F} = [F_1, F_2, \ldots, F_{10}]^T, \\
\boldsymbol{f}_{\mathrm{B}} = [(\Delta t)^2 y_n^{(2)}, (\Delta t)^3 y_n^{(3)}, \ldots, (\Delta t)^{p-3} y_n^{(p-3)}]^T,
\end{gathered}
$$

with the following N-vectors: Y_j, F_j for $j = 1, 2, \ldots, 10$, $y_n^{(j)}$ for $j = 2, 3, \ldots, p-3$, $Y_1 = y_n$, $F_1 = f_n$, $Y_{10} = y_{n+1}$ and $F_{10} = f_{n+1}$.

Thus, we can compactly write the HBT(p) method (20) in the following Shu–Osher form:

$$
\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{v} y_n^T + \boldsymbol{\alpha} \boldsymbol{Y} + \Delta t \boldsymbol{\beta} \boldsymbol{F} + \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}, \\
y_{n+1} &= Y_{10}.
\end{aligned}
\tag{25}
$$

Here consistency requires that

$$\boldsymbol{v} + \boldsymbol{\alpha} \boldsymbol{e}_{10} = \boldsymbol{e}_{10},$$

where the 10-vector $\boldsymbol{e}_{10}$ is

$$\boldsymbol{e}_{10} = [1, 1, 1, \ldots, 1]^T \in \mathbb{R}^{10}. \tag{26}$$

B. Butcher form in vector notation

This subsection describes a generalized result for the new HBT(p), using the result for RK methods, following closely section 3.2.1 of [4, pp. 31–32].

If $\boldsymbol{\alpha} = \boldsymbol{0}$, then the Shu–Osher form (25) becomes,

$$
\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{v} y_n^T + \Delta t \boldsymbol{\beta} \boldsymbol{F} + \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}, \\
y_{n+1} &= Y_{10}.
\end{aligned}
\tag{27}
$$

which is the Butcher form. The elements $\boldsymbol{v}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ of (27) are then denoted as $\boldsymbol{v}_0$, $\boldsymbol{\beta}_0$, $\boldsymbol{\gamma}_0$, respectively. Here, consistency requires that

$$\boldsymbol{v}_0 = \boldsymbol{e}_{10} \tag{28}$$

where $\boldsymbol{e}_{10}$ is defined in (26). Hence the Butcher form (27) can be rewritten as

$$
\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{e}_{10} y_n^T + \Delta t \boldsymbol{\beta}_0 \boldsymbol{F} + \boldsymbol{\gamma}_0 \boldsymbol{f}_{\mathrm{B}}, \\
y_{n+1} &= Y_{10}.
\end{aligned}
\tag{29}
$$

To find the relation between the Shu–Osher coefficients and the Butcher coefficients, we can solve (25) for $\boldsymbol{Y}$ since $\boldsymbol{I} - \boldsymbol{\alpha}$ is invertible because $\boldsymbol{\alpha}$ is strictly lower triangular,

$$\boldsymbol{Y} = (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{v} y_n^T + \Delta t (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\beta} \boldsymbol{F} + (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\delta} \boldsymbol{f}_{\mathrm{B}}. \tag{30}$$

Comparing (30) with (29), we have the following relations between the generalized Shu–Osher coefficients and the Butcher coefficients,

$$
\begin{aligned}
\boldsymbol{e}_{10} &= (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{v}, \\
\boldsymbol{\beta}_0 &= (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\beta}, \\
\boldsymbol{\gamma}_0 &= (\boldsymbol{I} - \boldsymbol{\alpha})^{-1} \boldsymbol{\delta}.
\end{aligned}
\tag{31}
$$

These relations will enable us to transform a Shu–Osher form of HBT(p) into its Butcher form and vice versa.

In fact, the form (29) is the Butcher form (10) and (11) with $\boldsymbol{\gamma}_0$ defined in (12) and the following matrix,

$$\beta_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 & \cdots & 0 & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ a_{9,1} & a_{9,2} & a_{9,3} & \cdots & a_{9,8} & 0 & 0 \\ b_1 & b_2 & b_3 & \cdots & b_8 & b_9 & 0 \end{bmatrix}. \qquad (32)$$

C. Canonical Shu–Osher form and formulation of the optimization problem in vector notation

To find the SSP coefficient of an HBT(p) method, it is useful to consider a particular Shu–Osher form of the matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, in which the ratio $r = \frac{\alpha_{i,j}}{\beta_{i,j}}$ is the same for i = 2, 3, 4, . . . , 10 and j = 1, 2, 3, . . . , i − 1, such that $\beta_{i,j} \neq 0$. Generally, if this particular Shu–Osher form of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is sparse, Shu–Osher forms of HBT(p) will allow for a reduced-storage implementation, just like optimal explicit SSP Runge–Kutta methods [10].

We shall denote the coefficient matrices of this special form by $\boldsymbol{\alpha}_r$, $\boldsymbol{\beta}_r$, and require that $\boldsymbol{\alpha}_r = r\boldsymbol{\beta}_r$. Substituting this relation into (31), we can solve for $\boldsymbol{\beta}_r$ in terms of $\boldsymbol{\beta}_0$ and r. Thus we find

$$\left(\boldsymbol{I} - r\boldsymbol{\beta}_r\right)^{-1}\boldsymbol{\beta}_r = \boldsymbol{\beta}_0 \quad \Leftrightarrow \quad \boldsymbol{\beta}_r = \boldsymbol{\beta}_0 - r\boldsymbol{\beta}_r\boldsymbol{\beta}_0$$
$$\Leftrightarrow \quad \boldsymbol{\beta}_r\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right) = \boldsymbol{\beta}_0.$$

Hence, since $\boldsymbol{I} + r\boldsymbol{\beta}_0$ is invertible, the coefficients for this form are given by

$$\boldsymbol{v}_r = \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{v}_0 \qquad = \left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right)\boldsymbol{v}_0, \qquad (33)$$

$$\boldsymbol{\beta}_r = \boldsymbol{\beta}_0\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \qquad = \boldsymbol{\beta}_0\left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right), \qquad (34)$$

$$\boldsymbol{\alpha}_r = r\boldsymbol{\beta}_r = r\boldsymbol{\beta}_0\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \quad = r\boldsymbol{\beta}_0\left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right), \qquad (35)$$

$$\boldsymbol{\delta}_r = \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{\gamma}_0 \qquad = \left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right)\boldsymbol{\gamma}_0,$$
(36)

where the identity $\left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right) = \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}$ follows from

$$\left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right)\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right) = \left(\boldsymbol{I} - r\boldsymbol{\beta}_r\right)\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)$$
$$= \boldsymbol{I} + r\boldsymbol{\beta}_0 - r\boldsymbol{\beta}_r - r^2\boldsymbol{\beta}_r\boldsymbol{\beta}_0 = \boldsymbol{I}$$

since $r\boldsymbol{\beta}_r = r\boldsymbol{\beta}_0 - r^2\boldsymbol{\beta}_r\boldsymbol{\beta}_0$.

It is to be noted that using (31) and (34), we can write $\boldsymbol{\beta}_r$ as

$$\begin{aligned} \boldsymbol{\beta}_r &= \boldsymbol{\beta}_0\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1} \\ &= \boldsymbol{\beta}_0\left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right) \\ &= \left(\boldsymbol{I} - \boldsymbol{\alpha}_r\right)\boldsymbol{\beta}_0 \\ &= \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{\beta}_0. \end{aligned} \qquad (37)$$

As in [4], we shall refer to the form obtained by the relations (33)–(36) as the *canonical Shu–Osher form* of HBT(p):

$$\boldsymbol{Y} = \boldsymbol{v}_r y_n^T + \boldsymbol{\alpha}_r\boldsymbol{Y} + \Delta t\boldsymbol{\beta}_r\boldsymbol{F} + \boldsymbol{\delta}_r\boldsymbol{f}_{\mathrm{B}}, \qquad (38)$$

which can be written in terms of the Butcher array:

$$\begin{aligned} \boldsymbol{Y} = {}& \left[\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{v}_0 y_n^T\right] + \left[r\boldsymbol{\beta}_0\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{Y}\right. \\ & \left. + \Delta t\boldsymbol{\beta}_0\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{F}\right] + \left[\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{\gamma}_0\boldsymbol{f}_{\mathrm{B}}\right]. \end{aligned} \qquad (39)$$

Using (37) and (39), we obtain

$$\boldsymbol{Y} = \left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\left[\boldsymbol{v}_0 y_n^T + \boldsymbol{\beta}_0\left(r\boldsymbol{Y} + \Delta t\boldsymbol{F}\right) + \boldsymbol{\gamma}_0\boldsymbol{f}_{\mathrm{B}}\right].$$
(40)

Here the consistency condition is

$$\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{v}_0 + r\left(\boldsymbol{I} + r\boldsymbol{\beta}_0\right)^{-1}\boldsymbol{\beta}_0\boldsymbol{e}_{10} = \boldsymbol{e}_{10}, \qquad (41)$$

which is equivalent to condition (28).

Generally, the sparse canonical Shu–Osher forms (38) or (40) will allow for reduced-storage implementation.

Note also that the Butcher form (29), with coefficient vector $\boldsymbol{v}_0$ and coefficient matrices $\boldsymbol{\beta}_0, \boldsymbol{\gamma}_0$, corresponds to the canonical Shu–Osher form (38) or (40) with r = 0.

The relations (33)–(36) will enable us to transform simply a Butcher form of an HBT(p) method into its canonical Shu–Osher form and vice versa.

Since the ratio $r = \frac{\alpha_{i,j}}{\beta_{i,j}}$ is the same for i = 2, 3, . . . , 10 and j= 1, 2, . . . , i − 1, the following slight modification of the result presented in Theorem 1 holds.

Theorem 2: If f satisfies conditions (3) of the forward Euler and (6) of the series S(p − 3) methods, then the one-step, 9-stage HBT(p) methods (20) satisfy the monotonicity property

$$\|y_{n+1}\| \leq y_n\|$$

provided

$$\Delta t \leq c(\boldsymbol{v}_r, \boldsymbol{\alpha}_r, \boldsymbol{\beta}_r, \boldsymbol{\delta}_r)\Delta t_{\mathrm{FE,S}(p-3)},$$

where

- the feasible SSP coefficient $c(\boldsymbol{v}_r, \boldsymbol{\alpha}_r, \boldsymbol{\beta}_r, \boldsymbol{\delta}_r)$ is the number:

$$r = \left\{\frac{\alpha_{i,j}}{\beta_{i,j}}\right\}, \quad i = 2, 3, \ldots, 10, \; j = 1, 2, \ldots, i-1, \qquad (42)$$

- conditions (24), with c_{feasible} replaced by r, are imposed on $\delta_{i,m}$, $i = 2, 3, \ldots, 10$, $m = 2, 3, \ldots, p-3$; to enhance the performance of the optimization software these conditions can be rewritten as:

$$\delta_{i,m}r^m m! - \alpha_{i,1} \leq 0, \quad \begin{cases} i = 2, 3, \ldots, 10, \\ m = 2, 3, \ldots, p-3, \end{cases} \qquad (43)$$

with the convention that $a/0 = +\infty$, under the assumption that all coefficients of (21) are nonnegative.

To optimize HBT(*p*) and obtain *c*(HBT(*p*)), by Theorem 2, we maximize

$$\max_{\boldsymbol{v}_r,\boldsymbol{\alpha}_r,\boldsymbol{\beta}_r,\boldsymbol{\delta}_r} c(\boldsymbol{v}_r,\boldsymbol{\alpha}_r,\boldsymbol{\beta}_r,\boldsymbol{\delta}_r) = c(\mathrm{HBT}(p)).$$

Hence, the problem of optimizing HBT(*p*) can be formulated as

$$c(\mathrm{HBT}(p)) = \max r, \tag{44}$$

subject to the component-wise inequalities

$$(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\,\boldsymbol{v}_0 \geq 0, \tag{45}$$

$$(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\,\boldsymbol{\beta}_0 \geq 0, \tag{46}$$

$$(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\,\boldsymbol{\gamma}_0 \geq 0, \tag{47}$$

together with conditions (43) and order conditions (13)–(19) for order *p*.

Since the consistency condition (41) is satisfied, inequality (45) is equivalent to the following inequality,

$$r\boldsymbol{\beta}_0\,(\boldsymbol{I} + r\boldsymbol{\beta}_0)^{-1}\,\boldsymbol{e}_{10} \leq \boldsymbol{e}_{10}.$$

It is to be noted that each representation $(\boldsymbol{v}_r,\boldsymbol{\alpha}_r,\boldsymbol{\beta}_r,\boldsymbol{\delta}_r)$ which satisfies conditions (45)–(47) together with conditions (43) and the order conditions (13)–(19) for order *p*, will produce a feasible SSP coefficient c_{feasible} defined in Theorem 1 and a feasible SSP HBT(*p*) in the Shu–Osher form (21).

Definition 1: The *effective SSP coefficient* of an SSP method *M* is denoted by

$$c_{\text{eff}}(M) = \frac{c(M)}{l}, \tag{48}$$

where *l* is the number of function evaluations of *M* per time step and *c*(*M*) is the SSP coefficient of *M*.

The SSP coefficients, *c*(HM(*k*, *p*)) and *c*(RK(*k*, *p*)), of hybrid and RK methods are found in [2], [8] and [18], respectively. In this paper, *l* = 5 + *p* for HBT(*p*), *l* = 2 for HM(*k*, *p*), and *l* = *k* for RK(*k*, *p*) (see [16]).

Gottlieb [3] pointed out that one looks for high-order SSP methods with $c(\cdot)$ as large as possible, taking their computational costs and orders into account. The coefficients $c_{\text{eff}}(\cdot)$ provide a fair comparison between methods of the same order, although, in practice, starting methods, storage issues and order reduction may also be important.

V. CONSTRUCTION OF HBT(9, P)

Since HBT(*p*) methods contain many free parameters, the MATLAB Optimization Toolbox was used to search for the methods with largest *c*(HBT(*p*)). Several authors, [18], [19], [4], have successfully used this technique to find optimal RK methods. In this work, the MATLAB Optimization Toolbox was used to tolerance 10^{-12} on the objective function *c*(HBT(*p*)) provided all the constraints were satisfied to tolerance 10^{-14}.

The formulae of new HBT(*p*) are listed in the Appendix with their *c*(HBT(*p*)), c_{eff}(HBT(*p*)) and abscissa vector σ.

A. Sixth-order methods

Ketcheson [12] pointed out that LM(*k*, 6) of order 6 with nonnegative coefficients requires at least *k* = 10 steps and $c_{\text{eff}}(\mathrm{LM}(10{,}6)) = 0.052$. Huang [8] introduced HM(*k*, 6) with *k*=5, 6, 7 and largest $c_{\text{eff}}(\mathrm{HM}(7{,}6)) = 0.220$. Ketcheson, Gottlieb and Macdonald [11] found a two-step 12-stage RK method of order 6 with $c_{\text{eff}}(\mathrm{TSRK}(12{,}6)) = 0.365$.

Our best HBT(6) has $c_{\text{eff}}(\mathrm{HBT}(6)) = 0.285$.

B. Seventh-order methods

Ketcheson [12] pointed out that LM(*k*, 7) of order 7 with nonnegative coefficients requires at least *k* = 12 steps with $c_{\text{eff}}(\mathrm{LM}(12{,}7)) = 0.018$. Huang [8] showed that HM(7,7) exists with $c_{\text{eff}}(\mathrm{HM}(7{,}7)) = 0.117$. Ketcheson, Gottlieb and Macdonald [11] found a two-step 12-stage RK methods of order 7 with $c_{\text{eff}}(\mathrm{TSRK}(12{,}7)) = 0.231$.

Our best HBT(7) has $c_{\text{eff}}(\mathrm{HBT}(7)) = 0.224$.

C. Higher-order methods

Ketcheson [12] pointed out that *k*-step LM(*k*, *p*) of order *p* with nonnegative coefficients require the indicated number of steps with corresponding c_{eff}:

$$c_{\text{eff}}(\mathrm{LM}(15{,}8)) = 0.012, \quad c_{\text{eff}}(\mathrm{LM}(18{,}9)) = 0.003,$$
$$c_{\text{eff}}(\mathrm{LM}(22{,}10)) = 0.010, \quad c_{\text{eff}}(\mathrm{LM}(26{,}11)) = 0.012,$$
$$c_{\text{eff}}(\mathrm{LM}(30{,}12)) = 0.002.$$

Two-step RK methods of order 8 with nonnegative coefficients are found in [11]. The best of these has

$$c_{\text{eff}}(\mathrm{TSRK}(12{,}8)) = 0.078.$$

We found HBT(8) with good $c_{\text{eff}}(\mathrm{HBT}(8)) = 0.188$.

It is not mentioned in the literature that general linear, one-step, multistage, SSP methods of order 9 to 12 with nonnegative coefficients exist. However, our study of one-step HBT(*p*) methods shows that SSP methods of high order with nonnegative coefficients exist.

For example, HBT(9, p) of order 9 to 12 have good effective SSP coefficients, shown in Table I. This table also lists $c(\mathrm{HBT}(p))$, $c_{\text{eff}}(\mathrm{HBT}(p))$, and $c(\mathrm{HM}(k,p))$, $c_{\text{eff}}(\mathrm{HM}(k,p))$ for the hybrid methods (HM) on hand. Column 8 lists $\mathrm{PEG}(c_{\text{eff}}(\mathrm{HBT}(p)), c_{\text{eff}}(\mathrm{HM}(k,p)))$ which is seen to be non negligible.

Table I also shows that the new HBT(*p*) methods are generally competitive with the other methods on hand. For example,

- $c_{\text{eff}}(\mathrm{HBT}(6)) > c_{\text{eff}}(\mathrm{HM}(k,6))$, for $k = 5, 6, 7$.
- $c_{\text{eff}}(\mathrm{HBT}(p)) > c_{\text{eff}}(\mathrm{HM}(7,7))$, for $p = 7, 8, 9, 10, 11$.

VI. NUMERICAL RESULTS

From now on, we shall use the total variation semi-norm,

$$\mathrm{TV}(y_n) = \sum_{j=1}^{N-1} |y_{n,j+1} - y_{n,j}|, \tag{49}$$

and say that a method is total variation diminishing (TVD) if

$$\mathrm{TV}(y_{n+1}) \leq \mathrm{TV}(y_n). \tag{50}$$

We compare our new methods numerically with HM of Huang [8].

A. Numerical verification of the order p of HBT(p)

To show the relevance of the theoretical order of HBT(*p*) when solving ODEs, we have applied these methods with various constant stepsizes on the following initial value problem over $t \in [0, t_n]$ with exact solution $y_i(t)$:

$$\begin{array}{lll} y_1' = -y_1, & y_1(0) = 1, & y_1(t) = e^{-t}, \\ y_2' = y_3, & y_2(0) = 0, & y_2(t) = \sin t, \\ y_3' = -y_2, & y_3(0) = 1, & y_3(t) = \cos t, \\ y_4' = 1, & y_4(0) = 0, & y_4(t) = t, \\ y_5' = -y_1 + (y_2 + y_4 y_3), & y_5(0) = 1, & y_5(t) = e^{-t} + t \sin t. \end{array} \tag{51}$$

In Fig. 1, the global error of y_2 and y_5 at t_n,

$$\max\{|y_{2,n} - y_2(t_n)|, |y_{5,n} - y_5(t_n)|\} = O(h^p),$$

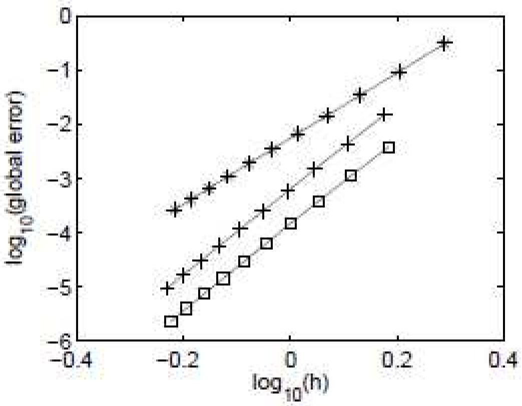
is plotted in a log-log scale for the listed HBT(*p*) methods applied to problem (51) over $t \in [0, t_n]$ with different constant stepsizes *h* so that the curves appear as straight lines with slope *p* whenever the leading term of the global error is of order *p*. For HBT(*p*), the slopes of the straight lines which approximate the data in the least-squares sense are very close to *p*, which confirms the orders of the methods.

It is to be noted that HBT(6), HBT(8), HBT(10), HBT(12) use $t_n = \pi + 32$ and HBT(7), HBT(9), HBT(11) use $t_n = 11\pi$.

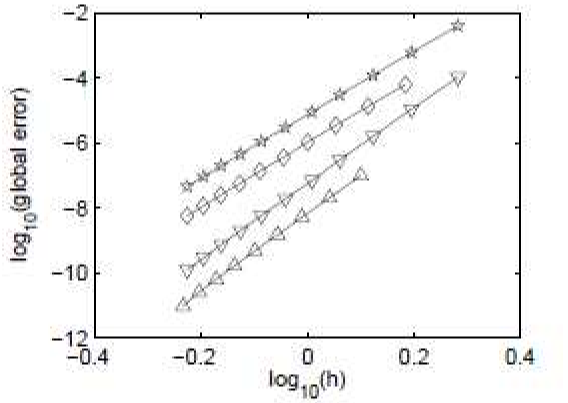
B. Percentage efficiency gain

Definition 2: The *percentage efficiency gain* (PEG) of the SSP coefficients $c_{\text{eff}}(\text{M2})$ of method 2 over $c_{\text{eff}}(\text{M1})$ of method 1 is

$$\mathrm{PEG}(c_{\text{eff}}(\text{M2}), c_{\text{eff}}(M1)) = \frac{c_{\text{eff}}(M2) - c_{\text{eff}}(M1)}{c_{\text{eff}}(M1)}. \tag{52}$$



HBT(6) ⋆, HBT(7) +, HBT(8) □



HBT(9) ⋆, HBT(10) ◇, HBT(11) ▽, HBT(12) △

Fig. 1. $\log_{10}(\text{global error})$ versus $\log_{10} h$ at t_n for the listed HBT(*p*) applied to problem (51) over $t \in [0, t_n]$ with constant stepsizes.

Table I

$\mathrm{PEG}(c_{\text{EFF}}(\mathrm{HBT}(k,p)), c_{\text{EFF}}(\mathrm{HM}(k,p)))$ TAKEN ROW-WISE.

p	HBT(p)	c(HBT(p))	c_{eff}(HBT(p))	HM(k,p)	c(HM(k,p))	c_{eff}(HM(k,p))	PEG
6	HBT(6)	3.134	0.285	HM(5,6)	0.209	0.104	173 %
	"	"	"	HM(6,6)	0.362	0.181	57 %
	"	"	"	HM(7,6)	0.440	0.220	30 %
7	HBT(7)	2.692	0.224	HM(7,7)	0.234	0.117	92 %
8	HBT(8)	2.449	0.188	"	"	"	61 %
9	HBT(9)	2.240	0.160	"	"	"	37 %
10	HBT(10)	2.057	0.137	"	"	"	17 %
11	HBT(11)	1.886	0.118	"	"	"	1 %
12	HBT(12)	1.782	0.105	"	"	"	-10 %

C. Comparing HBT(p) with other methods on Burgers' equation with a unit downstep initial condition

As a first comparison of our seven new methods with RK methods, following Huang [8], we consider Burgers' equation in Problem 1.

Problem 1: Burgers' equation with unit downstep initial condition,

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[\frac{1}{2}u(x,t)^2\right]=0, \qquad u(x,0)=\begin{cases}1, & x<0,\\ 0, & x\geq 0,\end{cases} \tag{53}$$

and boundary condition $u(-1,t)=1$ for $t\geq 0$.

We discretize the spatial derivative of the flux function $f(u)=u(x,t)^2/2$ by the weighted essentially non-oscillatory finite difference scheme of order 5 (WENO5) of Jiang and Shu [9] with spatial stepsize $\Delta x=1/150$. This leads to the semi-discrete system

$$\frac{d}{dt}u_j(t)=-\frac{1}{\Delta x}\left[f_{j+(1/2)}-f_{j-(1/2)}\right], \tag{54}$$

where $u_j(t) \approx u(x_j,t)$ with $x_j = j\Delta x$, $j = \ldots, -2, -1, 0, 1, 2, \ldots$, and $f_{j+(1/2)}$ is the numerical flux, which typically is a Lipschitz continuous function of several neighboring values $u_j(t)$ (see [9] for details). Then time discretization can be applied to (54).

We consider the total variation semi-norm (49) of the numerical solution at $t_{\text{final}}=1.8$. For this purpose, we let num_{eff} be the *largest effective CFL number* defined as

$$\text{num}_{\text{eff}}=\max_{\Delta t}\left\{\frac{\Delta t}{\Delta x}\frac{1}{l}\right\}, \tag{55}$$

such that the TV error in the numerical solution satisfies the inequality

$$|\text{TV}(u(x,t_{\text{final}}))-\text{TV}(u(x,0))|\leq 5.0\,\text{e-}02, \tag{56}$$

and we let $\max\Delta t_{\text{num}}=l\Delta x\,\text{num}_{\text{eff}}$ be the *maximum numerical time step*. Here *l* is the number of function evaluations per time step. We note that inequality (56) is used because t_{final} is small.

It was numerically observed that the TVD property (50) holds with error (56) for the methods listed in Table II with $\Delta t\leq\max\Delta t_{\text{num}}$. These numerical results confirm the result of Theorem 2 that HBT(*p*) methods are also TVD when combined with the WENO5 space discretization. The same situation holds for Problem 2 below.

Definition 3: The *percentage efficiency gain* of $\text{num}_{\text{eff}}(\text{M2})$ for method 2 over $\text{num}_{\text{eff}}(\text{M1})$ for method 1 is

$$\text{PEG}(\text{num}_{\text{eff}}(\text{M2}),\text{num}_{\text{eff}}(M1))=\frac{\text{num}_{\text{eff}}(\text{M2})-\text{num}_{\text{eff}}(\text{M1})}{\text{num}_{\text{eff}}(\text{M1})}. \tag{57}$$

We write $\text{PEG}(\text{num}_{\text{eff}})$ or more simply PEG when the context is clear.

For Problem 1, Table II lists num_{eff} in column 3 for HBT(*p*) and column 5 for OM(*k*, *p*) (other methods). The $\text{PEG}(\text{num}_{\text{eff}})$ is in column 8.

It is seen that:

(a) $\text{num}_{\text{eff}}(\text{HBT}(p))>\text{num}_{\text{eff}}(\text{OM}(k,p))$ for methods of the same order *p* and all *k*,

(b) quite remarkably, even though $c_{\text{eff}}(\text{HBT(12)})=0.105<c_{\text{eff}}(\text{HM}(7,7))=0.117$, in this example, HBT(12) allow a larger time step since $\text{num}_{\text{eff}}(\text{HBT}(12))=0.181>\text{num}_{\text{eff}}(\text{HM}(7,7))=0.127$,

(c) $\text{PEG}(\text{num}_{\text{eff}}(\text{HBT}(p)),\text{num}_{\text{eff}}(\text{OM}(k,p)))>0$ taken row-wise for all cases on hand.

A. D. Comparing HBT(p) and other methods on Burgers' equation with a square-wave initial condition

As a second comparison, we consider Burgers' equation with a square-wave initial value in Problem 2, which is the fourth of Laney's five test problems [13, p. 312].

Problem 2: Burgers' equation with a square wave initial condition,

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\left[\frac{1}{2}u(x,t)^2\right]=0, \qquad u(x,0)=\begin{cases}1, & |x|\leq\frac{1}{3},\\ 0, & \frac{1}{3}<|x|\leq 1,\end{cases} \tag{58}$$

and boundary condition $u(-1,t)=u(1,t)$ for $t\geq 0$. We discretize the spatial derivative of Problem 2 by WENO5 and compute the total variation of the numerical solution as a function of the effective CFL number (55) at $t_{\text{final}}=0.6$.

For Problem 2, Table III lists num_{eff} in columns 3 and 4 for HBT(*p*) and in columns 6 and 7 for OM(*k*, *p*) (other methods), respectively. The $\text{PEG}(\text{num}_{\text{eff}})$ is in column 8.

It is seen that the results for Problem 2 listed in Table III confirm the observations (a)–(c) obtained for Problem 1 as listed in Table II.

E. Differences between computing the higher derivatives directly or recurrently

In this section we describe the differences between computing the higher derivatives directly or using recurrences.

- The first approach is to compute directly the higher derivatives of the solution of (1) which, by successive differentiation, are obtained as

$$\begin{aligned} y^{(2)} &= f_t + f_y y' = f_t + f_y f, \\ y^{(3)} &= f_{tt} + 2f_{ty}f + f_{yy}f^2 + f_y(f_t + f_y f), \end{aligned} \tag{59}$$

etc. Then the solution is

$$y(t_0 + h) = y(t_0) + y^{(1)}(t_0)h + y^{(2)}(t_0)\frac{h^2}{2!} + \dots \tag{60}$$

Formulae (59) soon become very complicated for higher derivatives.

- The second approach, the "right approach", is, in fact, an extension of Newton's approach and has been rediscovered several times (Steffensen 1956 [20]). Let

$$Y^{[i]} = \frac{1}{i!}y^{(i)}(t_0), \quad F^{[i]} = \frac{1}{i!}f^{(i)}(t, y(t))|_{t=t_0} \tag{61}$$

be the Taylor coefficients of $y(t)$ and $f(t, y(t))$, so that (60) becomes

$$y(t_0 + h) = \sum_{i=0}^{\infty} h^i Y^{[i]}.$$

Then, from (1),

$$Y^{[i+1]} = \frac{1}{i+1}F^{[i]}. \tag{62}$$

Now suppose that $f(t, y)$ is the composition of a sequence of algebraic operations and elementary functions. This leads to a sequence of series [6, pp. 46–49],

$$y, p, q, r, \dots, \text{ and finally } f. \tag{63}$$

For each series $p = \sum_{i=0}^{\infty} P_i h^i$, $q = \sum_{i=0}^{\infty} Q_i h^i$, $r = \sum_{i=0}^{\infty} R_i h^i$, $\dots$, we find formulae for generating the *i*th Taylor coefficient from the preceding ones as follows:

a) $r = p \pm q$:

$$R_i = P_i \pm Q_i, \quad i = 0, 1, \dots \tag{64}$$

b) $r = pq$: the Cauchy product yields

$$R_i = \sum_{j=0}^{i} P_j Q_{i-j}, \quad i = 0, 1, \dots \tag{65}$$

c) $r = p/q$: write $p = rq$, use formula b) and solve for R_i:

$$R_i = \frac{1}{Q_0}\left[P_i - \sum_{j=0}^{i-1} R_j Q_{i-j}\right], \quad i = 0, 1, \dots \tag{66}$$

There are formulae for several other elementary functions.

VII. CONCLUSION

New explicit, 9-stage, one-step, SSP Hermite–Birkhoff–Taylor methods, called HBT(*p*), of orders *p* = 6, 7, . . . , 12, with nonnegative coefficients are constructed by combining Taylor methods with a 9-stage Runge–Kutta (RK(9,4)) method of order 4. We found no counterparts of one-step HBT(*p*) methods of order greater than 8 in the literature among hybrid and general linear, one-step, multistage methods. Our new HBT(*p*) tend to have larger effective SSP coefficients compared to effective SSP coefficients of hybrid methods [8] of the same order and other frequently used methods.

Table II

$\mathrm{PEG}(\mathrm{NUM_{EFF}}(\mathrm{HBT}(p)), \mathrm{NUM_{EFF}}(\mathrm{OM}(k,p))$ TAKEN ROW-WISE FOR PROBLEM 1.

p	$\mathrm{HBT}(p)$	$\mathrm{num_{eff}}(\mathrm{HBT}(p))$	$\mathrm{OM}(k,p)$	$\mathrm{num_{eff}}(\mathrm{OM}(k,p))$	PEG
6	HBT(6)	0.276	HM(5,6)	0.174	59 %
	"	"	HM(6,6)	0.169	63 %
	"	"	HM(7,6)	0.189	46 %
7	HBT(7)	0.271	HM(7,7)	0.127	113 %
8	HBT(8)	0.236	"	"	86 %
9	HBT(9)	0.221	"	"	74 %
10	HBT(10)	0.201	"	"	58 %
11	HBT(11)	0.194	"	"	53 %
12	HBT(12)	0.181	"	"	43 %

Table III

$\mathrm{PEG}(\mathrm{NUM_{EFF}}(\mathrm{HBT}(p)), \mathrm{NUM_{EFF}}(\mathrm{OM}(k,p))$ TAKEN ROW-WISE FOR PROBLEM 2.

p	$\mathrm{HBT}(p)$	$\mathrm{num_{eff}}(\mathrm{HBT}(p))$	$\mathrm{OM}(k,p)$	$\mathrm{num_{eff}}(\mathrm{OM}(k,p))$	PEG
6	HBT(6)	0.286	HM5,6	0.179	60 %
	"	"	HM(6,6)	0.174	64 %
	"	"	HM(7,6)	0.194	47 %
7	HBT(7)	0.271	HM(7,7)	0.124	119 %
8	HBT(8)	0.231	"	"	86 %
9	HBT(9)	0.226	"	"	82 %
10	HBT(10)	0.206	"	"	66 %
11	HBT(11)	0.179	"	"	44 %
12	HBT(12)	0.186	"	"	50 %

APPENDIX

This appendix lists the canonical Shu–Osher form of our seven new HBT(p) methods with their $c(\mathrm{HBT}(p))$, $c_{\mathrm{eff}}(\mathrm{HBT}(p))$ and abscissa vector σ for $p = 6, 7, \dots, 12$.

HBT(6). Here $c(\mathrm{HBT}(6)) = 3.1344502778184653$, $c_{\mathrm{eff}}(\mathrm{HBT}(6)) = 0.285$, and
$\sigma = [0, 0.31903520916464706, 0.44263770797105217, 0.57330852491969619, 0.63808367321401160, 0.63639798742340936, 0.71419617334655516, 0.81006609300994925, 0.99305760092443984]^T$

$$
\begin{aligned}
Y_2 &= y_n + 3.1903520916464706\,\text{e-01}\,\Delta t f_n + 5.0891732343365040\,\text{e-02}\,\Delta t^2 y_n^{(2)} \\
&\quad + 5.4120848243055667\,\text{e-03}\,\Delta t^3 y_n^{(3)}, \\
Y_3 &= 3.5834980295049018\,\text{e-01}\,y_n + 3.3219698318597529\,\text{e-02}\,\Delta t f_n \\
&\quad + 6.4165019704950976\,\text{e-01}\,Y_2 + 2.0470900482622731\,\text{e-01}\,\Delta t F_2 \\
&\quad + 5.6353599845246610\,\text{e-04}\,\Delta t^3 y_n^{(3)}, \\
Y_4 &= 3.1289996006153575\,\text{e-01}\,y_n + 4.9963033135710655\,\text{e-02}\,\Delta t f_n \\
&\quad + 6.8710003993846425\,\text{e-01}\,Y_3 + 2.1920910495880525\,\text{e-01}\,\Delta t F_3, \\
Y_5 &= 3.8396839112373654\,\text{e-01}\,y_n + 1.2249943597477428\,\text{e-01}\,\Delta t f_n \\
&\quad + 1.3421663550519683\,\text{e-01}\,Y_2 + 4.2819832381775655\,\text{e-02}\,\Delta t F_2 \\
&\quad + 4.8181497337106666\,\text{e-01}\,Y_4 + 1.5371594080809708\,\text{e-01}\,\Delta t F_4 \\
&\quad + 1.5775061778371173\,\text{e-02}\,\Delta t^2 y_n^{(2)}, \\
Y_6 &= 3.3508530924998969\,\text{e-01}\,y_n + 1.0690401172457090\,\text{e-01}\,\Delta t f_n \\
&\quad + 3.3508523636842058\,\text{e-01}\,Y_2 + 1.0690398847278423\,\text{e-01}\,\Delta t F_2 \\
&\quad + 3.2982946438158973\,\text{e-01}\,Y_5 + 1.0522720896729187\,\text{e-01}\,\Delta t F_5 \\
&\quad + 1.7053071870544181\,\text{e-02}\,\Delta t^2 y_n^{(2)}, \\
Y_7 &= 2.8522543163918057\,\text{e-01}\,y_n + 9.0996955242082775\,\text{e-02}\,\Delta t f_n \\
&\quad + 1.8817623450365853\,\text{e-01}\,Y_2 + 6.0034844334690363\,\text{e-02}\,\Delta t F_2 \\
&\quad + 5.2659833385716093\,\text{e-01}\,Y_6 + 1.6800340958787394\,\text{e-01}\,\Delta t F_6 \\
&\quad + 1.2754417488317356\,\text{e-02}\,\Delta t^2 y_n^{(2)}, \\
Y_8 &= 1.6949424914657354\,\text{e-01}\,y_n + 4.8507393639725199\,\text{e-02}\,\Delta t f_n \\
&\quad + 2.4432045297652949\,\text{e-01}\,Y_2 + 7.7946826818568390\,\text{e-02}\,\Delta t F_2 \\
&\quad + 5.8618529787689710\,\text{e-01}\,Y_7 + 1.8701374911739679\,\text{e-01}\,\Delta t F_7 \\
&\quad + 7.7377832379408179\,\text{e-03}\,\Delta t^2 y_n^{(2)} + 1.9747426853874376\,\text{e-05}\,\Delta t^3 y_n^{(3)}, \\
Y_9 &= 1.5934282872526689\,\text{e-01}\,y_n + 4.3870494155726060\,\text{e-02}\,\Delta t f_n \\
&\quad + 8.4065717127473327\,\text{e-01}\,Y_8 + 2.6819923647339500\,\text{e-01}\,\Delta t F_8 \\
&\quad + 7.4421514878145797\,\text{e-04}\,\Delta t^3 y_n^{(3)}, \\
y_{n+1} &= 1.1936675368070886\,\text{e-01}\,y_n + 3.2503556099446967\,\text{e-02}\,\Delta t f_n \\
&\quad + 1.4685688822062951\,\text{e-01}\,Y_4 + 4.6852518050737722\,\text{e-02}\,\Delta t F_4 \\
&\quad + 4.5303168990474829\,\text{e-01}\,Y_7 + 1.4453305994697488\,\text{e-01}\,\Delta t F_7 \\
&\quad + 2.8074468819391353\,\text{e-01}\,Y_9 + 8.9567433939104629\,\text{e-02}\,\Delta t F_9 \\
&\quad + 2.8634222716111407\,\text{e-03}\,\Delta t^2 y_n^{(2)},
\end{aligned}
$$

HBT(7). Here $c(\mathrm{HBT}(7)) = 2.6917712270543279$, $c_{\mathrm{eff}}(\mathrm{HBT}(7)) = 0.224$, and
$\sigma = [0, 0.37150259648711859, 0.49116343636981546, 0.47633474377755014, 0.58646639708358639, 0.70439249483623845, 0.80034165154884707, 0.86843685505824209, 0.92811365412780844]^T$

$$
\begin{aligned}
Y_2 &= y_n + 3.7150259648711859\,\text{e-01}\,\Delta t f_n + 6.9007089598335417\,\text{e-02}\,\Delta t^2 y_n^{(2)} \\
&\quad + 8.5454376539336133\,\text{e-03}\,\Delta t^3 y_n^{(3)} + 7.9366306913878205\,\text{e-04}\,\Delta t^4 y_n^{(4)}, \\
Y_3 &= 4.2225986856371001\,\text{e-01}\,y_n + 6.1899518523033714\,\text{e-02}\,\Delta t f_n \\
&\quad + 5.7774013143628999\,\text{e-01}\,Y_2 + 2.1463195892339088\,\text{e-01}\,\Delta t F_2 \\
&\quad + 1.0162655695684433\,\text{e-03}\,\Delta t^2 y_n^{(2)} + 1.3223961908677325\,\text{e-04}\,\Delta t^4 y_n^{(4)}, \\
Y_4 &= 7.8014066749875466\,\text{e-01}\,y_n + 2.8672956562212731\,\text{e-01}\,\Delta t f_n \\
&\quad + 2.1985933250124529\,\text{e-01}\,Y_3 + 8.1678312886137361\,\text{e-02}\,\Delta t F_3 \\
&\quad + 4.6838975201161356\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 3.8258666615004055\,\text{e-03}\,\Delta t^3 y_n^{(3)}, \\
Y_5 &= 4.4245107946678164\,\text{e-01}\,y_n + 1.1372215029603336\,\text{e-01}\,\Delta t f_n \\
&\quad + 5.5754892053321836\,\text{e-01}\,Y_4 + 2.0713087164668076\,\text{e-01}\,\Delta t F_4 \\
&\quad + 1.0026951030497215\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 6.7195732744727669\,\text{e-05}\,\Delta t^3 y_n^{(3)}, \\
Y_6 &= 4.1110237058195859\,\text{e-01}\,y_n + 1.4024682546646330\,\text{e-01}\,\Delta t f_n \\
&\quad + 5.8889762941804136\,\text{e-01}\,Y_5 + 2.1877699839391132\,\text{e-01}\,\Delta t F_5 \\
&\quad + 1.8505475311590643\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 8.2835777935509440\,\text{e-04}\,\Delta t^3 y_n^{(3)}, \\
Y_7 &= 3.0576829470769801\,\text{e-01}\,y_n + 1.1359371540734829\,\text{e-01}\,\Delta t f_n \\
&\quad + 1.8075810687540314\,\text{e-01}\,Y_2 + 6.7152106040308343\,\text{e-02}\,\Delta t F_2 \\
&\quad + 5.1347359841689888\,\text{e-01}\,Y_6 + 1.9075677503946195\,\text{e-01}\,\Delta t F_6 \\
&\quad + 2.1100180109224341\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 2.0306720589665624\,\text{e-03}\,\Delta t^3 y_n^{(3)}, \\
Y_8 &= 2.2124873183467206\,\text{e-01}\,y_n + 7.1947493821697545\,\text{e-02}\,\Delta t f_n \\
&\quad + 3.5834832878128309\,\text{e-01}\,Y_4 + 1.3312733458906631\,\text{e-01}\,\Delta t F_4 \\
&\quad + 4.2040293938404477\,\text{e-01}\,Y_7 + 1.5618078355198936\,\text{e-01}\,\Delta t F_7 \\
&\quad + 1.3364340382750778\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 1.6549623841765216\,\text{e-03}\,\Delta t^3 y_n^{(3)}, \\
Y_9 &= 1.8648977723784924\,\text{e-01}\,y_n + 4.8596460024534924\,\text{e-02}\,\Delta t f_n \\
&\quad + 4.5815517789662746\,\text{e-01}\,Y_5 + 1.7020583818261481\,\text{e-01}\,\Delta t F_5 \\
&\quad + 3.5535504486552338\,\text{e-01}\,Y_8 + 1.3201532184233844\,\text{e-01}\,\Delta t F_8 \\
&\quad + 3.4396158136530200\,\text{e-03}\,\Delta t^2 y_n^{(2)} + 1.0381950483538492\,\text{e-04}\,\Delta t^4 y_n^{(4)}, \\
y_{n+1} &= 1.8496595064455917\,\text{e-01}\,y_n + 6.4710509390639215\,\text{e-02}\,\Delta t f_n \\
&\quad + 2.7069734603786305\,\text{e-01}\,Y_5 + 1.0056476691523814\,\text{e-01}\,\Delta t F_5 \\
&\quad + 1.4061700321100362\,\text{e-01}\,Y_6 + 5.2239581803125337\,\text{e-02}\,\Delta t F_6 \\
&\quad + 4.0371970010657421\,\text{e-01}\,Y_9 + 1.4998291684259316\,\text{e-01}\,\Delta t F_9 \\
&\quad + 9.7056344207954517\,\text{e-03}\,\Delta t^2 y_n^{(2)} + 7.3019963420806666\,\text{e-04}\,\Delta t^3 y_n^{(3)},
\end{aligned}
$$

HBT(8). Here $c(\mathrm{HBT}(8)) = 2.4490190242034697$, $c_{\mathrm{eff}}(\mathrm{HBT}(8)) = 0.188$, and
$\sigma = [0, 0.40832675863971485, 0.51509463876233885, 0.54119230928400319, 0.60290489914617706, 0.71592709031537216, 0.83112256874134260, 0.88028605552884842, 0.94022587416946767]^T$

$$
\begin{aligned}
Y_2 &= y_n + 4.0832675863971485\,\text{e-01}\,\Delta t f_n + 8.3365370910607972\,\text{e-02}\,\Delta t^2 y_n^{(2)} \\
&\quad + 1.1346770562242042\,\text{e-02}\,\Delta t^3 y_n^{(3)} + 1.1582975111772071\,\text{e-03}\,\Delta t^4 y_n^{(4)} \\
&\quad + 9.4592773655887568\,\text{e-05}\,\Delta t^5 y_n^{(5)}, \\
Y_3 &= 4.9814578743579313\,\text{e-01}\,y_n + 1.0525363091028073\,\text{e-01}\,\Delta t f_n \\
&\quad + 5.0185421256420681\,\text{e-01}\,Y_2 + 2.0492050392602901\,\text{e-01}\,\Delta t F_2 \\
&\quad + 7.1494557204535708\,\text{e-03}\,\Delta t^2 y_n^{(2)} + 2.6685158277584848\,\text{e-05}\,\Delta t^4 y_n^{(4)} \\
&\quad + 1.7341087922815020\,\text{e-05}\,\Delta t^5 y_n^{(5)}, \\
Y_4 &= 7.4205604049085894\,\text{e-01}\,y_n + 3.0300133774265336\,\text{e-01}\,\Delta t f_n \\
&\quad + 2.5794395950914117\,\text{e-01}\,Y_3 + 1.0532542089706144\,\text{e-01}\,\Delta t F_3 \\
&\quad + 5.7972831778073483\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 6.5702517867146310\,\text{e-03}\,\Delta t^3 y_n^{(3)} \\
&\quad + 4.1867121796203288\,\text{e-04}\,\Delta t^4 y_n^{(4)}, \\
Y_5 &= 6.4046395861779915\,\text{e-01}\,y_n + 2.6151857224796643\,\text{e-01}\,\Delta t f_n \\
&\quad + 3.5953604138220091\,\text{e-01}\,Y_4 + 1.4680818639174846\,\text{e-01}\,\Delta t F_4 \\
&\quad + 4.9643600693644478\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 5.5278542749516921\,\text{e-03}\,\Delta t^3 y_n^{(3)} \\
&\quad + 3.4182473742704761\,\text{e-04}\,\Delta t^4 y_n^{(4)}, \\
Y_6 &= 4.6393705824681908\,\text{e-01}\,y_n + 1.7384327304872094\,\text{e-01}\,\Delta t f_n \\
&\quad + 5.3606294175318103\,\text{e-01}\,Y_5 + 2.1888884343294662\,\text{e-01}\,\Delta t F_5 \\
&\quad + 2.6878726697672974\,\text{e-02}\,\Delta t^2 y_n^{(2)} + 1.7959143417972294\,\text{e-03}\,\Delta t^3 y_n^{(3)} \\
&\quad + 6.4215973309279321\,\text{e-06}\,\Delta t^5 y_n^{(5)},
\end{aligned}
$$

$$\begin{aligned}
Y_7 &= 3.7063826519708932\,\mathrm{e}{-}01\, y_n + 1.5130068877991051\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 9.0648986111157317\,\mathrm{e}{-}02\, Y_2 + 3.6973573996881420\,\mathrm{e}{-}02\,\Delta t F_2\\
&+ 5.3891274869175354\,\mathrm{e}{-}01\, Y_6 + 2.2005249586292291\,\mathrm{e}{-}01\,\Delta t F_6\\
&+ 2.7084574180896895\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 2.2221744698585821\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
Y_8 &= 2.8437397615338866\,\mathrm{e}{-}01\, y_n + 1.0511682251854505\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 3.2385519735273466\,\mathrm{e}{-}01\, Y_4 + 1.3223874300366728\,\mathrm{e}{-}01\,\Delta t F_4\\
&+ 7.8510746076135410\,\mathrm{e}{-}02\, Y_5 + 3.2058038463654077\,\mathrm{e}{-}02\,\Delta t F_5\\
&+ 3.1326008041774134\,\mathrm{e}{-}01\, Y_7 + 1.2791247324819272\,\mathrm{e}{-}01\,\Delta t F_7\\
&+ 2.0355844552548158\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 2.9210342994010173\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 2.9818411683746175\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)},\\
Y_9 &= 2.3018771778819744\,\mathrm{e}{-}01\, y_n + 7.0200395777825858\,\mathrm{e}{-}02\,\Delta t f_n\\
&+ 4.3969999462606951\,\mathrm{e}{-}01\, Y_5 + 1.7954127357956298\,\mathrm{e}{-}01\,\Delta t F_5\\
&+ 3.3011228758573313\,\mathrm{e}{-}01\, Y_8 + 1.3479368037702377\,\mathrm{e}{-}01\,\Delta t F_8\\
&+ 7.2922218216523033\,\mathrm{e}{-}03\,\Delta t^2 y_n^{(2)} + 8.2800803038993870\,\mathrm{e}{-}05\,\Delta t^3 y_n^{(3)}\\
&+ 1.6262588742622116\,\mathrm{e}{-}05\,\Delta t^5 y_n^{(5)},\\
y_{n+1} &= 2.4198479236896717\,\mathrm{e}{-}01\, y_n + 9.8808865908124752\,\mathrm{e}{-}02\,\Delta t f_n\\
&+ 3.2554824625425346\,\mathrm{e}{-}02\, Y_2 + 1.3293006017384298\,\mathrm{e}{-}02\,\Delta t F_2\\
&+ 1.1468326025086042\,\mathrm{e}{-}01\, Y_5 + 4.6828243928468682\,\mathrm{e}{-}02\,\Delta t F_5\\
&+ 2.8993165794200371\,\mathrm{e}{-}01\, Y_6 + 1.1838685411449690\,\mathrm{e}{-}01\,\Delta t F_6\\
&+ 3.2084546481274334\,\mathrm{e}{-}01\, Y_9 + 1.3100978867124016\,\mathrm{e}{-}01\,\Delta t F_9\\
&+ 1.8726557061840815\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 2.0631207890874775\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 1.2597111208501440\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)},
\end{aligned}$$

HBT(9). Here $c(\mathrm{HBT}(9)) = 2.2400411250219729$, $c_{\mathrm{eff}}(\mathrm{HBT}(9)) = 0.160$, and $\sigma = [0, 0.44642037542511220, 0.54309646402715783, 0.57390482295733281, 0.63854657462054129, 0.73784712938817931, 0.83811359145139419, 0.88945480681336819, 0.94380092180103814]^T$

$$\begin{aligned}
Y_2 &= y_n + 4.4642037542511220\,\mathrm{e}{-}01\,\Delta t f_n + 9.9645575797349062\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)}\\
&+ 1.4827938452301360\,\mathrm{e}{-}02\,\Delta t^3 y_n^{(3)} + 1.6548734626642065\,\mathrm{e}{-}03\,\Delta t^4 y_n^{(4)}\\
&+ 1.4775384649672212\,\mathrm{e}{-}04\,\Delta t^5 y_n^{(5)} + 1.0993387937261849\,\mathrm{e}{-}05\,\Delta t^6 y_n^{(6)},\\
Y_3 &= 5.6191148975565075\,\mathrm{e}{-}01\, y_n + 1.5195318960173676\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 4.3808851024434919\,\mathrm{e}{-}01\, Y_2 + 1.9557163721271051\,\mathrm{e}{-}01\,\Delta t F_2\\
&+ 1.6516139058897837\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 7.1426032125819754\,\mathrm{e}{-}04\,\Delta t^3 y_n^{(3)}\\
&+ 5.3590697204572457\,\mathrm{e}{-}06\,\Delta t^5 y_n^{(5)} + 1.9267769413777157\,\mathrm{e}{-}06\,\Delta t^6 y_n^{(6)},\\
Y_4 &= 7.6526371284596395\,\mathrm{e}{-}01\, y_n + 3.4162931398791052\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 2.3473628715403611\,\mathrm{e}{-}01\, Y_3 + 1.0479106143720175\,\mathrm{e}{-}01\,\Delta t F_3\\
&+ 7.3153541642044490\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 9.7829316608283208\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 8.7148756332627864\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)} + 4.6540215386923776\,\mathrm{e}{-}05\,\Delta t^5 y_n^{(5)},
\end{aligned}$$

$$\begin{aligned}
Y_5 &= 6.6523160024094674\,\mathrm{e}{-}01\, y_n + 2.9697294072421165\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 3.3476839975905320\,\mathrm{e}{-}01\, Y_4 + 1.4944743470090061\,\mathrm{e}{-}01\,\Delta t F_4\\
&+ 6.2970832665343202\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 8.2353605224148828\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 7.0576977568103088\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)} + 3.5460999727557116\,\mathrm{e}{-}05\,\Delta t^5 y_n^{(5)},\\
Y_6 &= 5.4176896165858190\,\mathrm{e}{-}01\, y_n + 2.4068205553367641\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 4.5823103834141804\,\mathrm{e}{-}01\, Y_5 + 2.0456367216781485\,\mathrm{e}{-}01\,\Delta t F_5\\
&+ 4.8166300494901948\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 5.3609107235329731\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 2.9866382583543905\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)},\\
Y_7 &= 4.1931937817563186\,\mathrm{e}{-}01\, y_n + 1.8719271422819000\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 1.2613911858906621\,\mathrm{e}{-}01\, Y_2 + 5.6311072676319233\,\mathrm{e}{-}02\,\Delta t F_2\\
&+ 4.5454150323531201\,\mathrm{e}{-}01\, Y_6 + 2.0291658852060288\,\mathrm{e}{-}01\,\Delta t F_6\\
&+ 4.0057782576065799\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 4.9713077920957895\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 3.1659158178439276\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)},\\
Y_8 &= 3.3015276459569004\,\mathrm{e}{-}01\, y_n + 1.3010963431856629\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 3.8263814852393790\,\mathrm{e}{-}01\, Y_4 + 1.7081746591602623\,\mathrm{e}{-}01\,\Delta t F_4\\
&+ 2.8720908688037200\,\mathrm{e}{-}01\, Y_7 + 1.2821598839063936\,\mathrm{e}{-}01\,\Delta t F_7\\
&+ 2.6185485548769691\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 3.8809324141463927\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 4.8231441243178378\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)} + 4.2768652471481948\,\mathrm{e}{-}05\,\Delta t^5 y_n^{(5)},\\
Y_9 &= 2.9924826620465594\,\mathrm{e}{-}01\, y_n + 1.1309267353459040\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 4.2010650983273029\,\mathrm{e}{-}01\, Y_5 + 1.8754410583806108\,\mathrm{e}{-}01\,\Delta t F_5\\
&+ 2.8064522396261371\,\mathrm{e}{-}01\, Y_8 + 1.2528574824265473\,\mathrm{e}{-}01\,\Delta t F_8\\
&+ 1.7528005640333869\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 1.1797219779198561\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 4.8328406278208086\,\mathrm{e}{-}07\,\Delta t^5 y_n^{(5)} + 1.9442606319528250\,\mathrm{e}{-}06\,\Delta t^6 y_n^{(6)},\\
y_{n+1} &= 2.9794903273030865\,\mathrm{e}{-}01\, y_n + 1.3301051904901345\,\mathrm{e}{-}01\,\Delta t f_n\\
&+ 2.3425916167597982\,\mathrm{e}{-}02\, Y_2 + 1.0457806290216299\,\mathrm{e}{-}02\,\Delta t F_2\\
&+ 1.5748654207712234\,\mathrm{e}{-}01\, Y_5 + 7.0305201238471698\,\mathrm{e}{-}02\,\Delta t F_5\\
&+ 2.3933291466632498\,\mathrm{e}{-}01\, Y_6 + 1.0684308961692715\,\mathrm{e}{-}01\,\Delta t F_6\\
&+ 2.8180559435864594\,\mathrm{e}{-}01\, Y_9 + 1.2580375923048362\,\mathrm{e}{-}01\,\Delta t F_9\\
&+ 2.7770449850435419\,\mathrm{e}{-}02\,\Delta t^2 y_n^{(2)} + 3.4871335532053582\,\mathrm{e}{-}03\,\Delta t^3 y_n^{(3)}\\
&+ 2.7846790813182008\,\mathrm{e}{-}04\,\Delta t^4 y_n^{(4)} + 1.2827931920427374\,\mathrm{e}{-}05\,\Delta t^5 y_n^{(5)},
\end{aligned}$$

HBT(10). Here $c(\mathrm{HBT}(10)) = 2.0574649603145816$, $c_{\mathrm{eff}}(\mathrm{HBT}(10)) = 0.137$, and $\sigma = [0, 0.48803500875519273, 0.57512846131656870, 0.60611936184125625, 0.66906594986417223, 0.75859751573461798, 0.84880614486771233, 0.89780638486737785, 0.94770517842341273]^T$

$$\begin{aligned}
Y_2 &= y_n + 4.8603500875519273\,\text{e-01}\Delta t f_n + 1.1811501486783015\,\text{e-01}\Delta t^2 y_n^{(2)} \\
&\quad + 1.9136010761801849\,\text{e-02}\Delta t^3 y_n^{(3)} + 2.3251927895379565\,\text{e-03}\Delta t^4 y_n^{(4)} \\
&\quad + 2.2602501956411840\,\text{e-04}\Delta t^5 y_n^{(5)} + 1.8309345393789816\,\text{e-05}\Delta t^6 y_n^{(6)} \\
&\quad + 1.2712832641103549\,\text{e-06}\Delta t^7 y_n^{(7)}, \\
Y_3 &= 6.1333462109999792\,\text{e-01} y_n + 1.9926263967858385\,\text{e-01}\Delta t f_n \\
&\quad + 3.8666537890000208\,\text{e-01} Y_2 + 1.8793291081899244\,\text{e-01}\Delta t F_2 \\
&\quad + 2.8373412575235187\,\text{e-02}\Delta t^2 y_n^{(2)} + 2.1092054321094856\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 6.3417668786743507\,\text{e-06}\Delta t^4 y_n^{(4)} + 7.0682584553704366\,\text{e-07}\Delta t^6 y_n^{(6)} \\
&\quad + 1.9725755435686247\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_4 &= 7.9120429406124382\,\text{e-01} y_n + 3.8455298599120269\,\text{e-01}\Delta t f_n \\
&\quad + 2.0879570593875618\,\text{e-01} Y_3 + 1.0148202276399002\,\text{e-01}\Delta t F_3 \\
&\quad + 9.0793176186587624\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.3708908354563270\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 1.4542345593649708\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.0960423130804107\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 5.1581721030082841\,\text{e-06}\Delta t^6 y_n^{(6)}, \\
Y_5 &= 6.9802822243960017\,\text{e-01} y_n + 3.3926615320480269\,\text{e-01}\Delta t f_n \\
&\quad + 3.0197177756039995\,\text{e-01} Y_4 + 1.4676885555039010\,\text{e-01}\Delta t F_4 \\
&\quad + 7.9395878958359964\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.1750784920432470\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 1.2043843103912164\,\text{e-03}\Delta t^4 y_n^{(4)} + 8.6039091296908964\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 3.7371788682180800\,\text{e-06}\Delta t^6 y_n^{(6)}, \\
Y_6 &= 5.8703515013111041\,\text{e-01} y_n + 2.8531963433358010\,\text{e-01}\Delta t f_n \\
&\quad + 2.0423937810562353\,\text{e-02} Y_2 + 9.9267487925721858\,\text{e-03}\Delta t F_2 \\
&\quad + 3.9254091205832720\,\text{e-01} Y_5 + 1.9078862562904042\,\text{e-01}\Delta t F_5 \\
&\quad + 6.4987479745338528\,\text{e-02}\Delta t^2 y_n^{(2)} + 8.8970701320045375\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 7.5983173968174531\,\text{e-04}\Delta t^4 y_n^{(4)} + 3.4232100916219053\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 3.2163506117543243\,\text{e-08}\Delta t^7 y_n^{(7)}, \\
Y_7 &= 4.7320548851713212\,\text{e-01} y_n + 2.2999443375442946\,\text{e-01}\Delta t f_n \\
&\quad + 1.3521254999242316\,\text{e-01} Y_2 + 6.5718032919379332\,\text{e-02}\Delta t F_2 \\
&\quad + 3.9158196149044483\,\text{e-01} Y_6 + 1.9032254208138391\,\text{e-01}\Delta t F_6 \\
&\quad + 5.5273958051554196\,\text{e-02}\Delta t^2 y_n^{(2)} + 8.3204342416732975\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 8.0551086645095282\,\text{e-04}\Delta t^4 y_n^{(4)} + 4.2318696377853559\,\text{e-05}\Delta t^5 y_n^{(5)}, \\
Y_8 &= 3.9208118107696038\,\text{e-01} y_n + 1.7291373957351436\,\text{e-01}\Delta t f_n \\
&\quad + 3.5676608007532956\,\text{e-01} Y_4 + 1.7340080485296860\,\text{e-01}\Delta t F_4 \\
&\quad + 2.5115273884771006\,\text{e-01} Y_7 + 1.2206902362473740\,\text{e-01}\Delta t F_7 \\
&\quad + 3.8304906228222039\,\text{e-02}\Delta t^2 y_n^{(2)} + 5.9491106613240555\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 7.5651135872738338\,\text{e-04}\Delta t^4 y_n^{(4)} + 8.0411556093675949\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 5.9638502263927509\,\text{e-06}\Delta t^6 y_n^{(6)}, \\
Y_9 &= 3.6691631717889633\,\text{e-01} y_n + 1.6132319919474292\,\text{e-01}\Delta t f_n \\
&\quad + 3.9217126908470984\,\text{e-01} Y_5 + 1.9060896620312201\,\text{e-01}\Delta t F_5 \\
&\quad + 2.4091241373639380\,\text{e-01} Y_8 + 1.1709186711960279\,\text{e-01}\Delta t F_8 \\
&\quad + 3.1544686365824609\,\text{e-02}\Delta t^2 y_n^{(2)} + 3.3748148323606564\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 1.7692205549146444\,\text{e-04}\Delta t^4 y_n^{(4)} + 1.6886964680967587\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
y_{n+1} &= 3.5753110633438606\,\text{e-01} y_n + 1.7377263439748708\,\text{e-01}\Delta t f_n \\
&\quad + 4.8048625002385680\,\text{e-02} Y_4 + 2.3353313873709498\,\text{e-02}\Delta t F_4 \\
&\quad + 1.3948383506101963\,\text{e-01} Y_5 + 6.7794026995090545\,\text{e-02}\Delta t F_5 \\
&\quad + 2.0955104986037892\,\text{e-01} Y_6 + 1.0184914635354911\,\text{e-01}\Delta t F_6 \\
&\quad + 2.4538538374182961\,\text{e-01} Y_9 + 1.1926588713535649\,\text{e-01}\Delta t F_9 \\
&\quad + 3.9658003972039405\,\text{e-02}\Delta t^2 y_n^{(2)} + 5.5347106598610859\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 5.1205753144832329\,\text{e-04}\Delta t^4 y_n^{(4)} + 3.1361565925864263\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 1.1243666938384288\,\text{e-06}\Delta t^6 y_n^{(6)},
\end{aligned}$$

HBT(11). Here $c(\text{HBT}(11)) = 1.8863694597834419$, $c_{\text{eff}}(\text{HBT}(11)) = 0.118$, and
$\sigma = [0, 0.50166671204572799, 0.60233585164735337, 0.65715493934167046,$
$0.68278729875751931, 0.76517004442333447, 0.84919061738789059,$
$0.90388744108800101, 0.95374798897431523]^T$

$$\begin{aligned}
Y_2 &= y_n + 5.0166671204572799\,\text{e-01}\Delta t f_n + 1.2583474498738567\,\text{e-01}\Delta t^2 y_n^{(2)} \\
&\quad + 2.1042367592978138\,\text{e-02}\Delta t^3 y_n^{(3)} + 2.6390638410067305\,\text{e-03}\Delta t^4 y_n^{(4)} \\
&\quad + 2.6478609599932327\,\text{e-04}\Delta t^5 y_n^{(5)} + 2.2139061695900829\,\text{e-05}\Delta t^6 y_n^{(6)} \\
&\quad + 1.5866328983942980\,\text{e-06}\Delta t^7 y_n^{(7)} + 9.9495113670131378\,\text{e-08}\Delta t^8 y_n^{(8)}, \\
Y_3 &= 6.0289050664552324\,\text{e-01} y_n + 1.9260401168344962\,\text{e-01}\Delta t f_n \\
&\quad + 3.9710949335447671\,\text{e-01} Y_2 + 2.1051522611061857\,\text{e-01}\Delta t F_2 \\
&\quad + 2.5825585943063729\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.5758385541281734\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 8.8469167250134137\,\text{e-06}\Delta t^4 y_n^{(4)} + 1.7953038646240812\,\text{e-06}\Delta t^6 y_n^{(6)} \\
&\quad + 4.1675166072157112\,\text{e-07}\Delta t^7 y_n^{(7)} + 5.6202742607046147\,\text{e-08}\Delta t^8 y_n^{(8)}, \\
Y_4 &= 7.5035030110887069\,\text{e-01} y_n + 3.7443796521836487\,\text{e-01}\Delta t f_n \\
&\quad + 2.4964969889112934\,\text{e-01} Y_3 + 1.3234401012821184\,\text{e-01}\Delta t F_3 \\
&\quad + 9.0923251433214058\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.4198484266115499\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 1.5812247773282766\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.3051181548569144\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 7.8596555789745232\,\text{e-06}\Delta t^6 y_n^{(6)} + 2.9829064278184090\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_5 &= 7.4847197066513149\,\text{e-01} y_n + 3.9677909689603286\,\text{e-01}\Delta t f_n \\
&\quad + 8.1192214437425234\,\text{e-02} Y_2 + 4.3041522972252860\,\text{e-02}\Delta t F_2 \\
&\quad + 1.7033581489744326\,\text{e-01} Y_4 + 9.0298225522055506\,\text{e-02}\Delta t F_4 \\
&\quad + 1.0617003836077363\,\text{e-01}\Delta t^2 y_n^{(2)} + 1.8373328079696616\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 2.3412649390460432\,\text{e-03}\Delta t^4 y_n^{(4)} + 2.2591393685843274\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 1.6257323580695821\,\text{e-05}\Delta t^6 y_n^{(6)} + 7.5552647816939571\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_6 &= 6.5503844317538451\,\text{e-01} y_n + 3.4678405265594701\,\text{e-01}\Delta t f_n \\
&\quad + 3.4496155682461549\,\text{e-01} Y_5 + 1.8287062220791975\,\text{e-01}\Delta t F_5 \\
&\quad + 8.7470580910269907\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.3737912078463886\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 1.4573831432533521\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.0314096423626015\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 4.0580455494142769\,\text{e-06}\Delta t^6 y_n^{(6)}, \\
Y_7 &= 5.4594711224611481\,\text{e-01} y_n + 2.8941685278810125\,\text{e-01}\Delta t f_n \\
&\quad + 1.0761114874630667\,\text{e-01} Y_2 + 5.7046697924520363\,\text{e-02}\Delta t F_2 \\
&\quad + 3.4644173900757858\,\text{e-01} Y_6 + 1.8365529467771949\,\text{e-01}\Delta t F_6 \\
&\quad + 7.6456916615893919\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.2988073778997267\,\text{e-02}\Delta t^3 y_n^{(3)} \\
&\quad + 1.5221060359318086\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.2052499463397804\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 5.3428417539432416\,\text{e-06}\Delta t^6 y_n^{(6)}, \\
Y_8 &= 4.4655841035189719\,\text{e-01} y_n + 2.1773523413590418\,\text{e-01}\Delta t f_n \\
&\quad + 1.3069384505576909\,\text{e-01} Y_2 + 6.9283270240588460\,\text{e-02}\Delta t F_2 \\
&\quad + 1.9107659371398297\,\text{e-01} Y_5 + 1.0129330324077598\,\text{e-01}\Delta t F_5 \\
&\quad + 2.3167115087835072\,\text{e-01} Y_7 + 1.2281324301388283\,\text{e-01}\Delta t F_7 \\
&\quad + 5.5777976102390658\,\text{e-02}\Delta t^2 y_n^{(2)} + 9.9377426648108291\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 1.3515658227632177\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.4329810268645870\,\text{e-04}\Delta t^5 y_n^{(5)} \\
&\quad + 1.1445292014381784\,\text{e-05}\Delta t^6 y_n^{(6)} + 5.8504434047533321\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_9 &= 4.1553312770919232\,\text{e-01} y_n + 1.9602443252520632\,\text{e-01}\Delta t f_n \\
&\quad + 3.6366151467266078\,\text{e-01} Y_5 + 1.9278382227117358\,\text{e-01}\Delta t F_5 \\
&\quad + 2.2080635761814690\,\text{e-01} Y_8 + 1.1705308123653342\,\text{e-01}\Delta t F_8 \\
&\quad + 4.2414863135465525\,\text{e-02}\Delta t^2 y_n^{(2)} + 5.3690046048754016\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 4.0731930396135647\,\text{e-04}\Delta t^4 y_n^{(4)} + 1.5050838936146594\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 5.1089107693721366\,\text{e-08}\Delta t^7 y_n^{(7)} + 1.9391854202869462\,\text{e-08}\Delta t^8 y_n^{(8)}, \\
y_{n+1} &= 4.1851916683897278\,\text{e-01} y_n + 2.2186489749840380\,\text{e-01}\Delta t f_n \\
&\quad + 2.6754305683022249\,\text{e-02} Y_2 + 1.4182961637904000\,\text{e-02}\Delta t F_2 \\
&\quad + 8.1152443538902191\,\text{e-02} Y_5 + 4.3020439669447685\,\text{e-02}\Delta t F_5 \\
&\quad + 2.6843424332953464\,\text{e-01} Y_6 + 1.4230205113708280\,\text{e-01}\Delta t F_6 \\
&\quad + 2.0613984060956822\,\text{e-01} Y_9 + 1.0874849544750294\,\text{e-01}\Delta t F_9 \\
&\quad + 5.8740599490369484\,\text{e-02}\Delta t^2 y_n^{(2)} + 9.1621685388524627\,\text{e-03}\Delta t^3 y_n^{(3)} \\
&\quad + 1.0243344737203490\,\text{e-03}\Delta t^4 y_n^{(4)} + 8.1281745076304415\,\text{e-05}\Delta t^5 y_n^{(5)} \\
&\quad + 4.4328365430523062\,\text{e-06}\Delta t^6 y_n^{(6)} + 1.4161811040205456\,\text{e-07}\Delta t^7 y_n^{(7)},
\end{aligned}$$

HBT(12). Here $c(\text{HBT}(12)) = 1.7816072903992184$, $c_{\text{eff}}(\text{HBT}(12)) = 0.105$, and
$\sigma = [0, 0.51484115004650077, 0.61364261962237943, 0.68112578118031031,$
$0.70063387113308495, 0.77637248877477405, 0.85389110812929681,$
$0.91010350260719253, 0.95680184182362604]^T$

$$
\begin{aligned}
Y_2 = {} & y_n + 5.1484115004650077\,\text{e-01}\Delta t f_n + 1.3253070489060165\,\text{e-01}\Delta t^2 y_n^{(2)} \\
& + 2.2744086840783544\,\text{e-02}\Delta t^3 y_n^{(3)} + 2.9273979564666125\,\text{e-03}\Delta t^4 y_n^{(4)} \\
& + 3.0142898611020815\,\text{e-04}\Delta t^5 y_n^{(5)} + 2.5864674311054929\,\text{e-05}\Delta t^6 y_n^{(6)} \\
& + 1.9023140954116614\,\text{e-06}\Delta t^7 y_n^{(7)} + 1.2242369707892536\,\text{e-07}\Delta t^8 y_n^{(8)} \\
& + 7.0031952218953410\,\text{e-09}\Delta t^9 y_n^{(9)}, \\
Y_3 = {} & 6.2711319325693271\,\text{e-01} y_n + 2.1236716730481370\,\text{e-01}\Delta t f_n \\
& + 3.7288680674306729\,\text{e-01} Y_2 + 2.0929797983679765\,\text{e-01}\Delta t F_2 \\
& + 3.1104468324857616\,\text{e-02}\Delta t^2 y_n^{(2)} + 2.2925523352740184\,\text{e-03}\Delta t^3 y_n^{(3)} \\
& + 5.6261058932013941\,\text{e-05}\Delta t^4 y_n^{(4)} + 1.4253679899238594\,\text{e-06}\Delta t^6 y_n^{(6)} \\
& + 3.7819671857610225\,\text{e-07}\Delta t^7 y_n^{(7)} + 5.4858248826227185\,\text{e-08}\Delta t^8 y_n^{(8)} \\
& + 5.7653873650055428\,\text{e-09}\Delta t^9 y_n^{(9)}, \\
Y_4 = {} & 7.2298592751039970\,\text{e-01} y_n + 3.5566265570408539\,\text{e-01}\Delta t f_n \\
& + 2.7701407248960014\,\text{e-01} Y_3 + 1.5548548436144280\,\text{e-01}\Delta t F_3 \\
& + 8.4397814258881973\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.2723103891452371\,\text{e-02}\Delta t^3 y_n^{(3)} \\
& + 1.3433659445474212\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.0218177713752037\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 5.4005674692728717\,\text{e-06}\Delta t^6 y_n^{(6)} + 1.6320013282467639\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_5 = {} & 7.6762729627047777\,\text{e-01} y_n + 4.2661593485912302\,\text{e-01}\Delta t f_n \\
& + 8.8317196099774883\,\text{e-02} Y_2 + 4.9571640493222817\,\text{e-02}\Delta t F_2 \\
& + 1.4405550762974817\,\text{e-01} Y_4 + 8.0857048804211254\,\text{e-02}\Delta t F_4 \\
& + 1.1972782586771058\,\text{e-01}\Delta t^2 y_n^{(2)} + 2.2400713953238269\,\text{e-02}\Delta t^3 y_n^{(3)} \\
& + 3.1041329270093524\,\text{e-03}\Delta t^4 y_n^{(4)} + 3.3407994454647406\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 2.8305180624102512\,\text{e-05}\Delta t^6 y_n^{(6)} + 1.8361506636258773\,\text{e-06}\Delta t^7 y_n^{(7)} \\
& + 7.8392998735067049\,\text{e-08}\Delta t^8 y_n^{(8)}, \\
Y_6 = {} & 6.9294745138904945\,\text{e-01} y_n + 3.8894511440490743\,\text{e-01}\Delta t f_n \\
& + 2.6830839208939056\,\text{e-04} Y_2 + 1.5059906497647338\,\text{e-04}\Delta t F_2 \\
& + 3.0678424021886075\,\text{e-01} Y_5 + 1.7219520927651641\,\text{e-01}\Delta t F_5 \\
& + 1.0531990724383630\,\text{e-01}\Delta t^2 y_n^{(2)} + 1.8117789415324902\,\text{e-02}\Delta t^3 y_n^{(3)} \\
& + 2.1829729568626215\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.8948435983752427\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 1.1428402881549152\,\text{e-05}\Delta t^6 y_n^{(6)} + 3.9410370027418228\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_7 = {} & 5.8277459879513671\,\text{e-01} y_n + 3.2385650283682593\,\text{e-01}\Delta t f_n \\
& + 1.0733913722557595\,\text{e-01} Y_2 + 6.0248483380152561\,\text{e-02}\Delta t F_2 \\
& + 3.0988626397928720\,\text{e-01} Y_6 + 1.7393634705538763\,\text{e-01}\Delta t F_6 \\
& + 9.0888857657381988\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.6760577832828274\,\text{e-02}\Delta t^3 y_n^{(3)} \\
& + 2.2097586223279188\,\text{e-03}\Delta t^4 y_n^{(4)} + 2.1277249786162624\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 1.4337002283570911\,\text{e-05}\Delta t^6 y_n^{(6)} + 5.5418018307031255\,\text{e-07}\Delta t^7 y_n^{(7)}, \\
Y_8 = {} & 4.4471373248098967\,\text{e-01} y_n + 2.0914922797782209\,\text{e-01}\Delta t f_n \\
& + 1.8364685867274461\,\text{e-01} Y_2 + 1.0307931476391380\,\text{e-01}\Delta t F_2 \\
& + 1.4753896040245576\,\text{e-01} Y_5 + 8.2812279225348021\,\text{e-02}\Delta t F_5 \\
& + 2.2410044844382543\,\text{e-01} Y_7 + 1.2578554749493057\,\text{e-01}\Delta t F_7 \\
& + 5.3395901673811377\,\text{e-02}\Delta t^2 y_n^{(2)} + 9.9058800638159802\,\text{e-03}\Delta t^3 y_n^{(3)} \\
& + 1.4591094337997081\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.7299242835873391\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 1.6183179956936970\,\text{e-05}\Delta t^6 y_n^{(6)} + 1.1299196441342031\,\text{e-06}\Delta t^7 y_n^{(7)} \\
& + 4.9218345026036183\,\text{e-08}\Delta t^8 y_n^{(8)}, \\
Y_9 = {} & 4.5623472140780014\,\text{e-01} y_n + 2.2911065700082594\,\text{e-01}\Delta t f_n \\
& + 3.4564396895686095\,\text{e-01} Y_5 + 1.9400682227754573\,\text{e-01}\Delta t F_5 \\
& + 1.9812130963534624\,\text{e-01} Y_8 + 1.1120369270096088\,\text{e-01}\Delta t F_8 \\
& + 5.3713263821667148\,\text{e-02}\Delta t^2 y_n^{(2)} + 7.6104255475691086\,\text{e-03}\Delta t^3 y_n^{(3)} \\
& + 6.9401352289693869\,\text{e-04}\Delta t^4 y_n^{(4)} + 3.8400015003758447\,\text{e-05}\Delta t^5 y_n^{(5)} \\
& + 8.8722084831042739\,\text{e-07}\Delta t^6 y_n^{(6)} + 8.4609252643768281\,\text{e-09}\Delta t^8 y_n^{(8)} \\
& + 1.7976888708430029\,\text{e-09}\Delta t^9 y_n^{(9)}, \\
y_{n+1} = {} & 4.6243190037331180\,\text{e-01} y_n + 2.5955886174539222\,\text{e-01}\Delta t f_n \\
& + 2.8569609954456848\,\text{e-02} Y_2 + 1.6035862733843460\,\text{e-02}\Delta t F_2 \\
& + 5.5642225992119561\,\text{e-02} Y_5 + 3.1175347278509277\,\text{e-02}\Delta t F_5 \\
& + 2.7036747870107736\,\text{e-01} Y_6 + 1.5175481160076196\,\text{e-01}\Delta t F_6 \\
& + 1.8308872497904383\,\text{e-01} Y_9 + 1.0276603938796060\,\text{e-01}\Delta t F_9 \\
& + 7.1049057286457926\,\text{e-02}\Delta t^2 y_n^{(2)} + 1.2465455708880464\,\text{e-02}\Delta t^3 y_n^{(3)} \\
& + 1.5488442911868266\,\text{e-03}\Delta t^4 y_n^{(4)} + 1.4176580961226606\,\text{e-04}\Delta t^5 y_n^{(5)} \\
& + 9.5701957722357722\,\text{e-06}\Delta t^6 y_n^{(6)} + 4.5521534748983638\,\text{e-07}\Delta t^7 y_n^{(7)} \\
& + 1.2851999109827081\,\text{e-08}\Delta t^8 y_n^{(8)},
\end{aligned}
$$

To advance integration from t_n to t_{n+1}, once y_{n+1} is obtained by formula (21), the function *g*, with input (t_{n+1}, y_{n+1})

$$[f_{n+1}, y_{n+1}^{(2)}, \ldots, y_{n+1}^{(p-3)}] = g(t_{n+1}, y_{n+1}),$$

outputs f_{n+1} and $y_{n+1}^{(2)}$ to $y_{n+1}^{(p-3)}$ by means of the recurrent power series method. In adding, multiplying or taking powers of input power series, this method computes, in a recurrent way, the *k*th term of the output power series as a combination of the preceding terms of the input series.

For precision and efficiency, Horner's scheme is used to evaluate the summation

$$\sum_{m=2}^{p-3} (\Delta t)^m \frac{\delta_{i,m}}{\alpha_{i,1}} y_n^{(m)}$$

in (21) as nested polynomials in Δt.

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