Slovak University of Technology in Bratislava
Faculty of Mechanical Engineering
14th Conference on Applied Mathematics **APLIMAT 2015**

Proceedings

ON STOCHASTIC RESONANCE IN PIPELINE, INDUCED BY PULSED FLUID FLOW

CARKOVS Jevgeņijs (LV), MATVEJEVS Andrejs (LV)

Abstract. This paper deals with stability analysis of elastic pipes containing water flow, the velocity of which is perturbed harmonically with random rush of phas. Applying diffusion approximation approch to analysis of Markov dynamical systems we derive the Lyapunov index for the pipe shapes. We have proved that for the pipe safekeeping examination is not suffuiicient only to make analysis of the first shape dynamics but also we should examine the Lyapunov index for the shapes of more large numbers.

Key words: Pipeline dynamics, stochastic resonance, Markov dynamical systems.

Mathematics Subject Classification: Primary 60G99; Secondary 76E99.

1 Introduction

There is a large literature devoted to stability analysis of the wave propagation in periodic media (see, for example, the monographs [12], [14], and the extensive bibliography there). Analytical and numerical techniques thereto is mainly uses a study of the linearized equations applying linear operator spectral theory with further application of deterministic or probabilistic perturbation theory methods and results. Our paper as well deals with linearized partial differential equation

$$EJ\frac{\partial^4 u}{\partial x^4} + P(t)\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

developed in [3] for analysis of the transverse oscillations of a pipeline section of length L under the action of pulse fluid flow. Here we use the following notations:

- EJ – flexural rigidity of pipeline;
- $P(t)$ – disturbance longitudinal force, involved by fluid flow;
- m – mass of unit of pipeline length;
- D – dissipation factor.

This model has been sufficiently in detail studied in [8] for pin-ended pipeline section assuming boundary condition in a form of equalities

$$u(t,0) = u(t,L) = 0; \quad \left(\frac{\partial^2 u}{\partial x^2}\right)(t,0) = \left(\frac{\partial^2 u}{\partial x^2}\right)(t,L) \tag{2}$$

and periodic longitudinal force

$$P(t) = p + h\cos(\nu t + \varphi) \tag{3}$$

where is fluid oscillation frequency. The author of paper [8] decompose solution of equation 1-2 in Fourier series

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{4}$$

and discuss behavior of the first shape amplitude $T_1(t)$. Under assumption of sufficiently great mass and infinitesimality for dissipation factor they apply well known Bogolyubov-Mitropolsky method [2] to the second order ordinary differential equation Mathieu for function and discuss pipeline stability under initiation of resonance. It should be mentioned that the above periodic term (3) with constant phase arises in compliance with mathematical model for wave propagation in a form of the classical KDV equation [3] with constant coefficients. But in reality there are many interesting wave phenomena in fluids, which may be studied only by stochastic analysis approach (see, for example, [4]). The most advanced of the above indicated papers ([11], [10]) assume the index of refraction in wave equation as random function of an with zero mean and small variance. Under these assumptions the authors study behavior of the second moment for wave amplitude, quantify most of emergent random effects, and show that the perturbations to the phase of wave oscillation are asymptotic to Brownian motion. Therefore our paper deals with model of fluid-caused longitudinal force in (1) as random oscillation given by formula

$$P(t) = p + H\cos(y(t)) \tag{5}$$

where $y(t)$ satisfies the simplest stochastic differential equation [15]

$$dy(t) = \nu dt + \sigma dw(t) \tag{6}$$

on the circle [7] and $w(t)$ is the standard Brownian motion process. Applying decomposition (4) we derive the second order ordinary differential equations for shape amplitudes $T_n(t)$ with stochastic coefficients. As well as in the paper [8] we assume the mass m and the flexural regidity factor EJ sufficiently large. This makes it possible to take advantage of described in [5] asymptotical methods for stability analyzes of derived equations for the shape amplitudes $T_n(t), n \in \mathbb{N}$. In difference of results [8] we will prove that in presence of random phase (6) we should analyze not only the first shape $T_1(t)$. At every coefficient set in (1) there exists such a number $\hat{n}$ that amplitude $T_{\hat{n}}$ has the minimal Lyapunov index, which defines stability properties of equation (1). Besides stochastic behavior of phase in (5) permits to avoid resonant effect for equation (1).

2 Lyapunov indices for the shapes.

As in the paper [8] we apply decomposition (4) to solution of equation (1) with above (5)-(6) longitudinal force $P(t)$ and derive equations for shape amplitudes $T_n(t)$:

$$\begin{aligned} EJ\left(\frac{\pi n}{L}\right)^4 T_n(t) &- \left(\frac{\pi n}{L}\right)^2 (p + h\cos(\nu t + w(t))T_n(t) + \\ &+ DT_n'(t) + mT_n''(t) = 0, \ \ n \in \mathbf{N} \end{aligned} \tag{7}$$

where $\{w(t), t \geq 0\}$ is the Wiener process on the circle $\mathbb{Y} = \mathbb{R}\left(\mathrm{mod}\frac{2\pi}{\sigma}\right)$ [7]. It is well known that the density function $p(t,z,y) := \frac{\partial}{\partial y}\mathrm{P}(y(s+t) \leq y/y(s) = z)$ of this process satisfies Kolmogorov partial differential equation [6]

$$\frac{\partial}{\partial t}p(t,z,y) := \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}p(t,z,y) \tag{8}$$

with boundary condition

$$p(t,x,0) = p(t,x,2\pi), \ \left(\frac{\partial}{\partial z}p\right)(t,x,0) = \left(\frac{\partial}{\partial z}p\right)(t,x,2\pi) \tag{9}$$

It easy to make sure that linear operator $Q := \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}$ in the space $\mathbf{C}^2(\mathbf{Y})$ with periodic boundary condition (9) has the discrete spectrum consisting of the simple spectrum point 0 and the points $-\frac{\sigma^2}{2}k^2, k \in \mathbf{N}$.This means [7] that $\{w(t), t \geq 0\}$ is an ergodic Markov process with the limit uniform distribution on the segment $[0, 2\pi/\sigma]$. Let us denote $\{\hat{y}(t), t \geq 0\}$ given by the above infinitesimal generator Q stationary Markov function, that is the Wiener process on the circle $\mathbb{Y}$ which at any time moment t has the uniform distribution on the segment $[0, 2\pi/\sigma]$. This permits to state that a difference between $w(t)$ and $\hat{y}(t)$ with probability one tends to zero. Therefore for stability analysis of equation (7) we may substitute in this equation $\hat{y}(t)$ instead of $w(t)$. Now as in the paper [8] we assume mass m and flexural rigidity factor EJ sufficiently large and dissipation factor D sufficiently small we can introduce a small parameter $\varepsilon > 0$ and rewrite equation (7) in a following form

$$T_n''(t) + 2\varepsilon^2\delta T_n'(t) + \omega_n^2 T_n(t) - \varepsilon h_n \cos(y(t))T_n(t) = 0, n \in \mathbf{N} \tag{10}$$

where

$$2\varepsilon^2\delta = \frac{D}{m}, \omega_n^2 = \frac{EJ}{m}\left(\frac{\pi n}{L}\right)^4 + \left(\frac{\pi n}{L}\right)^2\frac{p}{m}, \varepsilon h_n = \left(\frac{\pi n}{L}\right)^2\frac{H}{m} \tag{11}$$

To discuss parametric resonance problem we should put $\nu = 2\omega_n$ in the definition (6) of Markov process $y(t)$. Substituting

$$T_n(t) = r(t)\sin(\omega_n t + \varphi(t)), \ T_n'(t) = -r(t)\omega_n \sin(\omega_n t + \varphi(t)) \tag{12}$$

one can write the equaqtions for the shape $T_n(t)$ oscillation amplitude $r(t)$ and phase $\varphi(t)$:

$$\begin{aligned} \dot{r}(t) &= \varepsilon F_1(t, r(t), \varphi(t), w(t)) + \varepsilon^2 F_2(t, r(t), \varphi(t), w(t)), & (13)\\ \dot{\varphi}(t) &= \varepsilon \Phi_1(t, r(t), \varphi(t), w(t)) + \varepsilon^2 \Phi_2(t, r(t), \varphi(t), w(t)) & (14) \end{aligned}$$

where

$$\begin{aligned} \Phi_1(t,\varphi,w) &= \frac{1}{2\omega_n}[g + h_n\cos(2\omega_n t + \sigma w) + g\cos(2\omega_n t + 2\varphi) + \\ &+ \frac{h_n}{2}(\cos(4\omega_n t + 2\varphi + \sigma w) + \cos(\sigma w - 2\varphi))], \\ \Phi_2(t,\varphi) &= -a\sin(2\omega_n t + 2\varphi), \\ F_1(t,r,\varphi,w) &= \frac{r}{2\omega_n}(g\sin(2\omega_n t + 2\varphi) + \frac{h_n}{2}(\sin(4\omega_n t + 2\varphi + \sigma w) - \sin(\sigma w - 2\varphi))), \\ F_2(t,r,\varphi) &= ar(\cos(2\omega_n t + 2\varphi) - 1). \end{aligned}$$

To find the Lyapunov index

$$\Lambda_n := \lim_{t\to\infty} \sup \frac{1}{t} \ln |T_n(t)| \tag{15}$$

for defined by (10) the shape jscillation T_n we will use well known diffusion approximation approach (see, for example, [13],[9]), applying defined the paper [5] the resolvent operator

$$(\mathcal{R}f)(t,\, y) := \int\limits_t^\infty (\exp\{Q(s-t)\}(f(s,y) - \widehat{f}(s))ds - \int\limits_0^t (\widehat{f}(s) - \widehat{\widehat{f}})ds \tag{16}$$

where $Q := \frac{\partial^2}{\partial y^2}$ is infinitesemal operator of the diffusion process $w(t)$ on the circle $\mathbb{Y}$, and

$$\hat{f}(s) := \frac{\sigma}{2\pi} \int\limits_0^{2\pi/\sigma} f(s,y)dy,\ \hat{\hat{f}} = \frac{\omega_n}{\pi} \int\limits_0^{\pi/\omega_n} \hat{f}(s)ds \tag{17}$$

The weak infinitesimal operator $L(\varepsilon)$ of the compound Markov process $\{r(t), \varphi(t), w(t)\}$ may be decomposed by the powers of the small parameter ε:

$$\begin{aligned} L(\varepsilon) &= L_0 + \varepsilon L_1 + \varepsilon^2 L_2, \\ L_0 &:= \frac{\partial}{\partial t} + Q,\ L_j := F_j \frac{\partial}{\partial r} + \Phi_j \frac{\partial}{\partial \varphi}, j = 1,2 \end{aligned}$$

By definition of diffusion approximation ([13], [9]) we should apply $L(\varepsilon)$ to function

$$v(t,r,\varphi,y,\varepsilon) := \varepsilon^{-2} v_0(r) + \varepsilon^{-1} v_1(t,r,\varphi,y) + v_2(t,r,\varphi,y)$$

where

$$\begin{aligned} v_1(t,r,\varphi,\xi) &= \mathcal{R}\left\{F_1 v_0'(r)\right\} \\ v_2(t,r,\varphi,\xi) &= \mathcal{R}\left\{\left(F_1 \frac{\partial}{\partial r} + \Phi_1 \frac{\partial}{\partial \varphi}\right) \mathcal{R}\left\{F_1 v_0'(r)\right\}\right\} + \mathcal{R}\left\{F_2 v_0'(r)\right\} \end{aligned}$$

and rush the parameter ε to zero. As a result we will have the equality:

$$\lim_{\varepsilon\to 0} v(t,r,\varphi,y,\varepsilon) = \{\mathcal{L}v_0\}(r)$$

where

$$\{\mathcal{L}v_0\}(r) = \widehat{\overline{\left[\left(F_1 \frac{\partial}{\partial r} + \Phi_1 \frac{\partial}{\partial \varphi}\right) \mathcal{R}\left\{F_1\right\} + F_2\right]}} v_0'(r) + \widehat{\overline{F_1 \mathcal{R}\left\{F_1\right\}}} v_0''(r) \tag{18}$$

Defined by this operator diffusion process satisfies stochastic differential Ito equation [15]

$$dr = mrdt + grdw(t) \tag{19}$$

and has the solution, which is called *the stochastic exponent*

$$r(t) = r(0) \exp\left\{\left(m - \frac{g^2}{2}\right) t + gw(t)\right\} \tag{20}$$

where

$$
\begin{aligned}
m &:= \frac{1}{r}\overline{\widehat{\left[\left(F_1\frac{\partial}{\partial r}+\Phi_1\frac{\partial}{\partial \varphi}\right)\mathcal{R}\left\{F_1\right\}+F_2\right]}} \\
g &:= \frac{1}{r}\sqrt{2\overline{\widehat{F_1\mathcal{R}\left\{F_1\right\}}}}
\end{aligned}
$$

The proportion $\frac{w(t)}{t}$ by the strong law of large numbers [6] with probability one tends to zero with $t \to \infty$. Therefore the Lyapunov index of the shape $T_n(t)$ may be given by formula

$$
\begin{aligned}
&\Lambda_n := \lim_{t\to\infty}\frac{1}{t}|T_n(t)| = \lim_{t\to\infty}\frac{1}{t}\ln r(t) = \\
&= \overline{\widehat{\left[\left(F_1\frac{\partial}{\partial r}+\Phi_1\frac{\partial}{\partial \varphi}\right)\mathcal{R}\left\{F_1\right\}+F_2\right]}} = \left[\frac{h_n^2}{8\omega_n^2\sigma^2}\left(\frac{8\omega_n^2+\sigma^4}{16\omega_n^2+\sigma^4}\right)-\delta\right]
\end{aligned}
\tag{21}
$$

Now we can substitute $\omega_n^2 = \frac{\lambda_n}{m}\left(EJ\lambda_n + p\right), h_n = \lambda_n\frac{H}{m}$, where $\lambda_n = \left(\frac{\pi n}{L}\right)^2$ in (21) and rewrite the formula (21) in a following form

$$
\Lambda_n = \Lambda(\lambda_n) = \left[\frac{H^2\lambda_n}{\left(\lambda_n+\rho\right)\mu\sigma^2}\left(\frac{\left(\lambda_n\left(\lambda_n+\rho\right)\right)+\mu\sigma^4}{2\left(\lambda_n\left(\lambda_n+\rho\right)\right)+\mu\sigma^4}\right)-\delta\right]
\tag{22}
$$

where $\mu := \frac{m}{8EJ}$ and $\rho := \frac{p}{EJ}$.

3 Dependence of the maximum of Lyapunov index on the pipe parameters

The authors of paper [8] prove that for a damage of the pipe (1) nonmetering the random perturbations the most dangerous is the parametric resonace for the first shape $T_1(t)$. It will be recalled that for $\sigma = 0$ the parametric resonace condition for the equation (10) is the equality $2\omega_n = \nu$, which has been taken into account in the equation (12).Therefore in the judgment of the above authors to analyse a stability of the beam we should deal only with the equation (10) with $\nu = 2\omega_1$. But as it has been shown in [1], [13], and [5] in a presence of a random phase in the periodic parametric perturbation of a form (3)-(6), we can avoid an effect of the parametric resonace. The assumption $\sigma \neq 0$ helps us to derive the formula (22) for the stability analysis of the shapes $T_n(t), n = 1, 2, ...$ even for $2\omega_n = \nu$. As it follows from the formula (22), a behavior of the amplitudes of the pipe oscillations defines the supremum of Lyapunov index $\Lambda(\lambda_n)$.First of all we can make sure that the function $\{\Lambda(\lambda_n), n = 1, 2, ...\}$ is bounded because

$$
\lim_{n\to\infty}\Lambda(\lambda_n) = \frac{1-2\delta}{2\mu\sigma^2}H^2
\tag{23}
$$

It is well known [1] the solutions of the linear differential equations (10) asymptotically decay to zero if and only if Lyapunov index is negative for any $n \in \mathbb{N}$. Therefore the necessary condition for beam oscillation decay is the equality

$$
\sup_{n\in\mathbb{N}}\Lambda(\lambda_n) < 0 \Leftrightarrow \delta > 0.5
\tag{24}
$$

So far as we are interested in the argument λ_{max} of the function (22) then we can deal with function

$$
\bar{\Lambda} = \frac{\lambda}{(\lambda+\rho)}\frac{\lambda(\lambda+\rho)+\sigma^4}{2\lambda(\lambda+\rho)+\sigma^4}
\tag{25}
$$

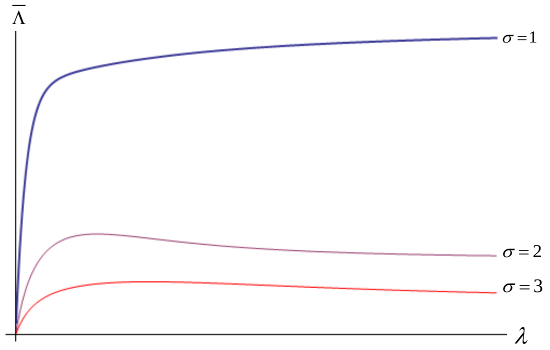

Fig. 1. Sketches of the Lyapunov index dependence on variance for $\rho = 1$.

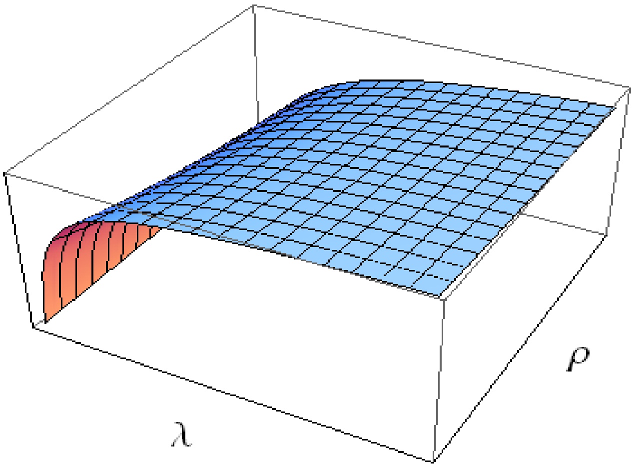

Fig. 2. Sketche of the Lyapunov index dependence on the parameters λ and ρ for $\mu = 1, \sigma = 0.01$.

For simplicity in the above formula we put $\mu = 1$. This does not imply to the final conclusion because we are looking for a tendency of Lyapunov index with increasing λ_n. Unfortunatelly we may not apply the above derived formula (22) for $\sigma = 0$ because at it is well known [3] in this case we have parametric resonance. But we can do that for sufficiently small $\sigma > 0$. The model in the neighborhood of resonance for sufficiently small $\varepsilon > 0$ has been analyzed in the paper [8]. The author show that the parametric resonance is more destructive for the first shape. As we can see from the diagrammatic drawings on Fig.1 the Lyapunpv index as a function of λ_n for some value of the other parameters has the pronounced maximum and stabilizes for increasing λ_n. Expressed on the Fig. 2 the monotone increasing function with incresing λ asymptotically approach the plane $\bar{\Lambda} = 0.5$. We may not apply [15] the diffusion approximation approach to the asymptotic analysis of equation (10) for very small σ. But the above result shows that for the pipe safekeeping analysis in a presence of random perturbations (even very small!) we may not be guided only by the deterministic equation for the first shape $T_1(t)$.

Now we put $\sigma = 1$ in the formula (25). It has been proved in many papers (see, for example, [1],[13], [5]) that for asymptotic stability analysis of the linear differential equations (10) one may apply the derived above formula (22). As we can see from the Fig. 3 below the maximum of the Lyapunov index for the some constant p in formula (3) may be reached by shapes with $n > 1$.

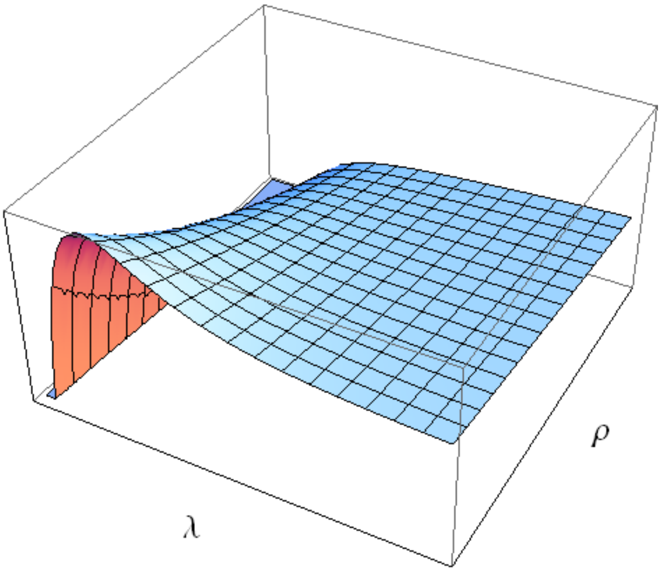

Fig. 3. Sketche of the Lyapunov index dependence on the parameters λ and ρ for $\mu = 1, \sigma = 1$.

This work was partially supported by the LZP project 623/2014.

References

[1] BLANKENSHIP, G., PAPANICOLAOU, G. Stability and control of stochastic systems with wide-band noise disturbances.I. In *SIAM J. Appl. Math.*, , Nr. 3, 1978: pp. 437–476.

[2] BOGOLJUBOV, N. N., MITROPOLSKIJ, J. A., SAMOILENKO, A. *Methods of Accelerated Convergence in Nonlinear Mechanics*. New York: Springer-Verlag, 1976.

[3] BOLOTIN, V. *Dinamicheskaya ustoichivost' uprugikh system (Dynamic stability of elastic systems)*. Moscow: Gostoptekhizdat, 1956.

[4] BOLOTIN, V. *Random vibration of elastic systems.* Berlin: Springer-Science-Business Media, B.V., 2010.

[5] CARKOVS, J., STOYANOV, J. Asymptotic methods for stabilitry analysis of Markov dynamical systems with fast variables. In *From Stochastic Analysis to Mathematical Finance. Festschrift for Professor Albert Shiryaev*, Berlin, 2005, pp. 93–110.

[6] DOOB, J. L. *Stochastic Processes*. New York: John Willey & Sons, 1953.

[7] DYNKIN, E. B. *Markov Processes*. Berlin: Springer-Verlag, 1965.

[8] ISHEGUZHIN, I., GABBASOV, I., SHAMMAZOV, M., et al. Demping parametrical vibrations of the pipeline. In *Oil and Gas. The electronic scientific journal*, 2011: pp. 84–93.

[9] KHASMINSKY, R. Limit theorem for a solution of the differential equation with a random right part. In *Prob. Theor. and its Applic.*, Volume 11, Nr. 3, 1966: pp. 260–279.

[10] MEI, C. Localization of nonlinear dispersive waves in weakly random media. In *Proc. Roy. Soc. Lond.*, 2002: pp. 119–134.

[11] MEI, C., PIHL, A. Resonant reflection of surface water waves by periodic sand bars. In *J . Fluid Mech.*, 1980: pp. 315–335.

[12] MURAVSKI, K. *Analytical and Numerical Methods for Wave Propagation in Fluid Media (Stability, Vibration and Control of Systems, Series A)*. World Scientific Pub. Co. Inc., 2003.

[13] PAPANICOLAOU, G., KELLERT, J. Stochastic differential equations with applications to random harmonic oscillators and wave propagation in random media. In *SIAM J. Appl. Math.*, , Nr. 2, 1971: pp. 287–305.

[14] ROMEO, F., RUZZENE, M. *Wave Propagation in Linear and Nonlinear Periodic Media: Analysis and Applications (CISM International Centre for Mechanical Sciences)*. New York: Springer, 2012.

[15] SKOROHOD, A. *Asymptotic methods of the theory of stochastic differential equations*. Provedence: AMS, 1994.

Current Address

Jevgeņijs Carkovs, dr. hab. math.
Riga Technical University
1 Kalku Street, Riga LV-1658, Latvia
Phone: +371 67089517, e-mail: jevgenijs.carkovs@rtu.lv

Andrejs Matvejevs, dr. sci. eng.
Riga Technical University
1 Kalku Street, Riga LV-1658, Latvia
Phone: +371 67089580, e-mail: andrejs.matvejevs@rtu.lv