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Different Approaches to Clustering – Cassini Ovals

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Abstract **– Classical cluster analysis or clustering is the task of grouping of a set of objects in such a way that objects in the same group are more similar to each other than to those in other groups or clusters. There are many clustering algorithms for solving different tasks. In the research, an interesting method – Cassini oval – has been identified. The ovals of Cassini are defined to be the sets of points in the plane for which the product of the distances to two fixed points is constants. Cassini ovals are named after the astronomer Giovanni Domenico Cassini who studied them in 1680. Cassini believed that the Sun travelled around the Earth on one of these ovals, with the Earth at one focus of the oval. Other names include Cassinian ovals. A family of military applications of increasing importance is detection of a mobile target intruding into a protected area potentially well suited for this type of application of Cassini style method. The hypothesis is proposed that the Cassini ovals could be used for clustering purposes. The main aim of the research is to ascertain the suitability of Cassini ovals for clustering purposes.**

Keywords **– Bistatic radar, Cassini ovals, clustering.**

I. Introduction

Ancient Greeks extolled the sphere, considering it to be a complete self-sufficient ideal form lying in the foundation of the universe (*Cult of the Sphere*). It was the idea of the motion of planets around the Sun that lied at the heart of the "Ptolemy astronomy". However, in the 17th century this age-old "Ptolemy idyll" was destroyed by new astronomical laws established by Johann Kepler. The most important of them is Kepler's First Law, according to which the motion of the planets corresponds not to the ideal circle, but to another geometric figure – an ellipse. It is known that an ellipse is a planar figure, for each point of which, the sum of the distances from two fixed points (poles) is constant. From the ratio of the distances between the foci and the sum of the distances, different figures can be obtained from a circle to degeneration into a line.

The study of ellipse, in which the sum of distances of each point from the two foci is constant, leads to an idea, what if not the sum of the distances from two points is constant but their product? The first who thought about this idea was Giovanni Cassini. In 1680, he began to study the curve called the Cassini ovals, which is the geometric place of points where product of the distances from two fixed foci is constant [6].

If we denote by a half of the distance between the foci of the oval, and b is the value of the product of the distances from the oval points to the foci, then it is possible to derive the following expression for the Cassini ovals:

$$[(x-a)^2+y^2][(x+a)^2+y^2]=b^4. \tag{1}$$

After opening the brackets and combining similar terms, the Cassini oval equation is obtained in the following form (in Cartesian coordinates):

$$(x^2+y^2+a^2)^2-4a^2x^2=b^4. \tag{2}$$

Cassini became interested in these curves, with the purely practical goals [2]–[4], [7], [11]. He came to these curves, trying to find the optimal mathematical model of the Earth's motion around the Sun. This way, he showed that the convex version of this curve for planetary orbits fit more than the ellipse proposed by Kepler.

II. Geometric Interpretation of Cassini Ovals

Equation (2) can be transformed into polar coordinates. Let $x=r\cos(\theta)$ and $y=r\sin(\theta)$. Then

$$r^4+a^4-2a^2r^2[1+\cos(2\theta)]=b^4. \tag{3}$$

Solving for r^2 using the quadratic equation gives:

$$r^2=\frac{2a^2\cos(2\theta)\pm\sqrt{4a^4\cos^2(2\theta)-4(a^4-b^4)}}{2}=$$
$$=a^2\cos(2\theta)\pm\sqrt{a^4\cos^2(2\theta)+b^4-a^4}=$$
$$=a^2\cos(2\theta)\pm\sqrt{a^4[\cos^2(2\theta)-1]+b^4}=$$
$$=a^2\cos(2\theta)\pm\sqrt{b^4-a^4\sin^2(2\theta)}==$$
$$a^2\left[\cos(2\theta)\pm\sqrt{\left(\frac{b}{a}\right)^4-\sin^2(2\theta)}\right]. \tag{4}$$

The geometric figures corresponding to the Cassini oval equation have the form shown in Fig. 1.

As follows from Fig. 1, Cassini ovals have four characteristic shapes that depend on the ratio between a and b. If $b\geq 2a$, then Cassini oval is a convex curve (Fig. 1a) similar to an ellipse. If $a<b<2a$, then a concave bridge appears in the Cassini oval (Fig. 1b). If $a=b$, then the bridge closes and the Cassini oval turns into a figure recalling the inverted digit 8 (Fig. 1c). This curve in mathematics is known as *lemniscat Bernoulli*, which can be defined as the geometric place of the points for which the product of the distances from two foci is equal to the square of half of the distance between the foci. Bernoulli, a great mathematician and physicist, described this "similar to 8 surface" in one of his articles published in 1694. Unfortunately, he did not know that his lemniscate was a particular case of ovals described by Cassini fourteen years earlier. Finally, at $a>b$, the Cassini oval splits into two independent figures (Fig. 1d).

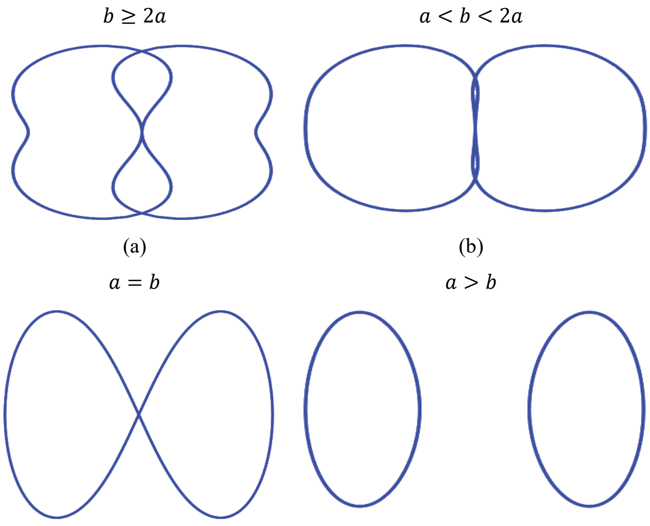

Fig. 1. Cassini ovals (four characteristics).

For the purpose of research, a program in the Matlab environment was developed. Results are demonstrated in Figs. 5–8.

III. The Use of Cassini Ovals in Radiolocation

In the Cartesian coordinates system (x, y) on the axis x symmetrically to zero, there are transmitter F_1 and receiver F_2 (see Fig. 2). Let us find the locus of the reflector M, for which its observability by the receiver F_2 is constant. Obviously, this is a set of points for which $1/{R_1}^2{R_2}^2$ or what is the same [5]

$$R_1 R_2 = const = \alpha^2. \quad (5)$$

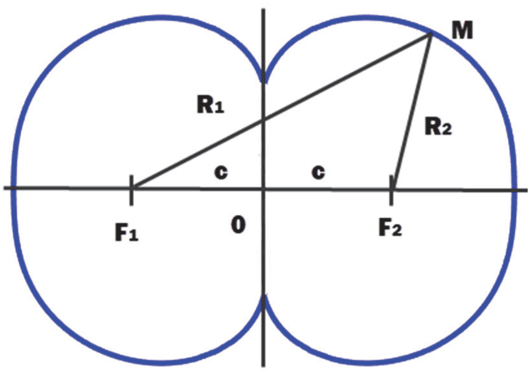

Fig. 2. Transmitter and receiver notations for Cassini oval.

Let us define $F_1O = OF_2 = c$ and express R_1 and R_2 in terms of c, x and y. From Fig. 2 it can be seen that

$${R_1}^2 = y^2 + (x + c)^2 \text{ and } {R_2}^2 = y^2 + (x - c)^2 . \quad (6)$$

Then

$${R_1}^2{R_2}^2 = a^4 = [y^2 + (x + c)^2][y^2 + (x - c)^2]. \quad (7)$$

After elementary transformations, the following equation is obtained:

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4. \quad (8)$$

This is the Cassini oval equation – the geometric locus of the points M for which the product of the distances R_1 and R_2 from the two foci F_1 and F_2 is constant.

For radiolocation Cassini ovals have the following meaning. From the target located at any point on the given oval, the same power will be taken into focus (if the transmitter that irradiates the target is in the second focus). If the power is at the sensitivity limit, then this oval covers the radar coverage area. In three-dimensional space, this is the surface formed by the rotation of the oval about the axis x. The larger is the number of Cassini oval on which the target is located, the higher is the power of the received signal [5].

The so-called bistatic radar employs two sites that are separated by a considerable distance. A transmitter is placed at one site, and the associated receiver is placed at the other site. Target detection is similar to that of monostatic radar: target illuminated by the transmitter and target echoes detected and processed by the receiver (see Fig. 3).

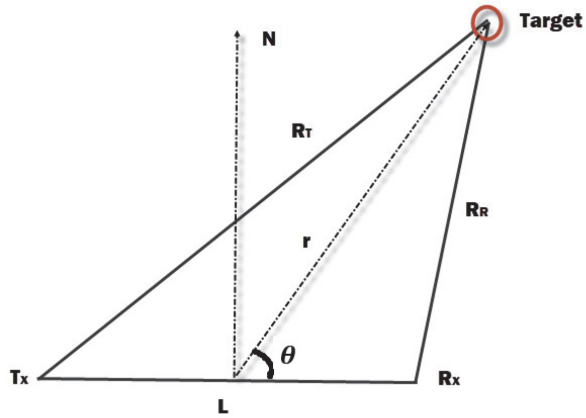

Fig. 3. Bistatic radar north coordinate system for two coordinates.

Cassini ovals define three distinct operating regions for a bistatic radar: receiver-centred region, transmitter-centred region, and receiver-transmitter-centred region.

The value of the bistatic radar constant k is critical to the selection of these operating regions. The term k is bistatic radar maximum range product [5].

Coverage is an important factor in bistatic radars. Coverage can be defined as the area on the bistatic plane whereby the target is visible to both the transmitter and the receiver [8].

Bistatic radar coverage is determined by both sensitivity and propagation. Propagation requires a suitable path between the target and the both sites and should include the effects of multipath, diffraction, refraction, shadowing, absorption and geometry.

The first five effects are usually included in the pattern propagation. The geometry effect is treated separately [8].

One type of coverage is constrained by the maximum range product of the Cassini oval $(R_T R_R)_{\max}$. When the Cassini oval encapsulates both the transmitter and the receiver, the coverage area can be approximated by [5], [8]:

$$S \approx \pi k \left\{1 - \left(\frac{1}{64}\right)\left(\frac{L^4}{k^2}\right) - \left(\frac{3}{16384}\right)\left(\frac{L^8}{k^4}\right)\right\}. \qquad (9)$$

IV. Clustering Possibilities Using Cassini Ovals

In general, clustering algorithms are used to group some given objects defined by a set of numerical properties in such a way that the objects within a group are more similar than the objects in different groups [1]. Therefore, a particular clustering algorithm needs to be given a criterion to measure the similarity of objects, how to cluster the objects into groups. The classical c-means clustering algorithm uses the Euclidean distance to measure the similarities between objects [1], [9], [10].

A sample of artificial data with 14 points [12] has been taken to test the Cassini oval possibilities for clustering purposes. The coordinates of the points are provided in Table I.

TABLE I

The Coordinates of the Experimental Data Points

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x	1	3	6	10	2	2	5	6	4	8	8	4	9	1
y	3	4	1	6	3	8	5	5	3	6	3	9	1	6

By means of the *c*-means clustering algorithm, the following clusters and their centres have been derived (see Fig. 4).

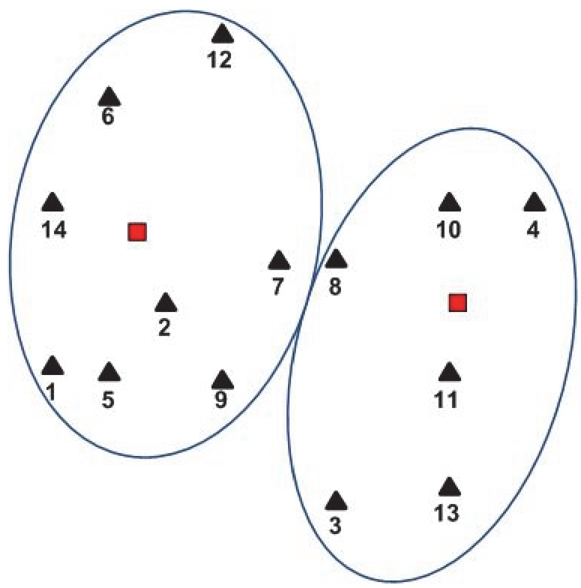

Fig. 4. The obtained two clusters with centres using *c*-means algorithm.

The values of ovals a and b have been changed in the study and it has been found whether the data points fit into the ovals.

a) Case $b \geq 2a$ (Fig. 1a)

Starting with $a = 5$ and $b = 10$, all points are inside the oval. Really one cluster has been obtained (see Fig. 5).

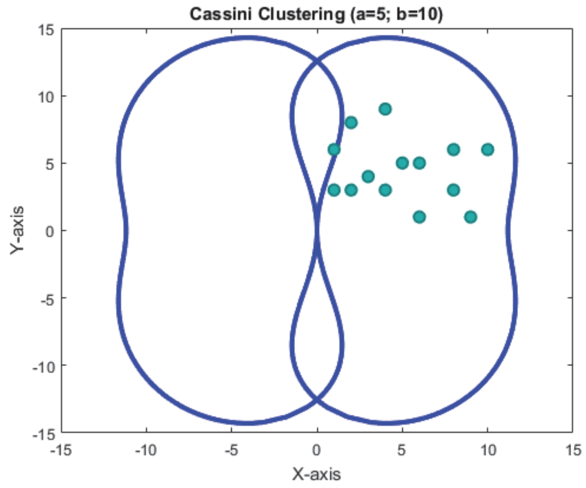

Fig. 5. Clustering for $b \geq 2a$.

b) Case $a < b < 2a$ (Fig. 1b)

Starting with a = 5 and b = 9, all points are inside the oval. One cluster has been obtained (see Fig. 6).

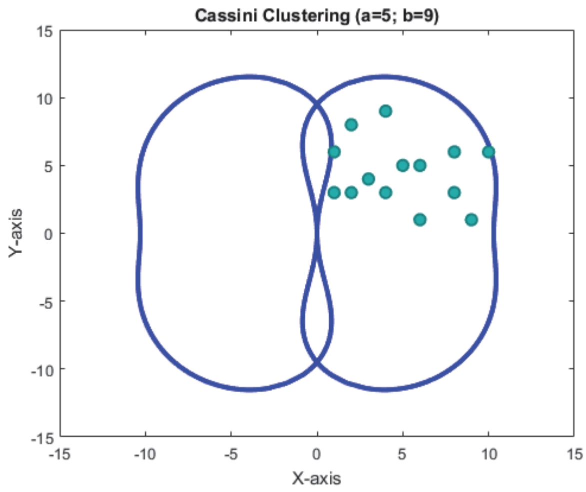

Fig. 6. Clustering for $a < b < 2a$.

c) Case $a = b$ (Fig. 1c)

For a and b values from 13 data points, only 7 data points are in the oval (see Fig. 7).

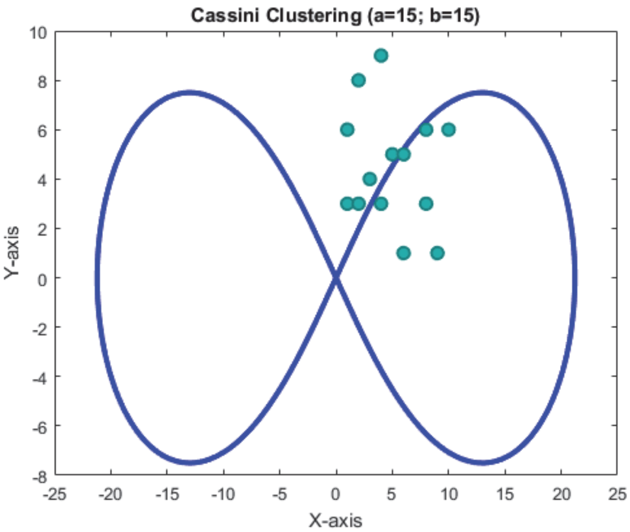

Fig. 7. Clustering for $a = b$.

d) Case $a > b$ (Fig. 1d)

At $a = 15$, $b = 14$ and $a = 16$, $b = 15$ only 3 data points are inside the oval. From $a = 17$, $b = 16$, 4 points are inside the oval. The value of b in all cases is $a - 1$. Data points do not appear inside the oval at the lower values of b (see Fig. 8).

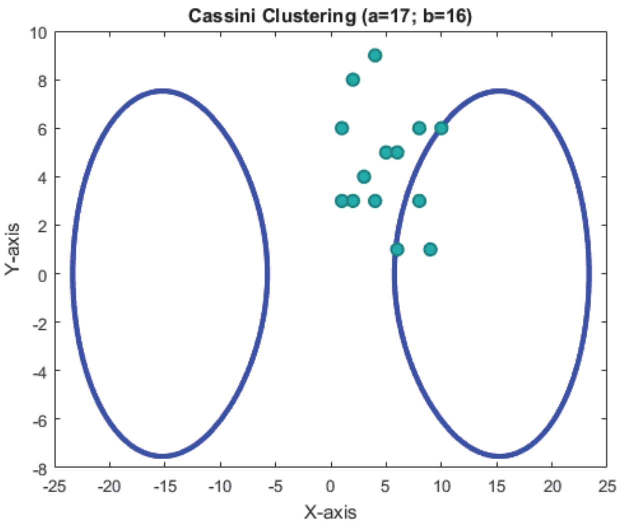

Fig. 8. Clustering for $a > b$.

Obviously, the cases c) and d) are more interesting. Both data points and ovals are currently set to the starting point of the coordinates. Making ovals "floating" and using more ovals, it is possible to cover all data points and obtain multiple clusters (see Fig. 9).

The results obtained so far do not allow them to be compared with the results of the classical c-means algorithm.

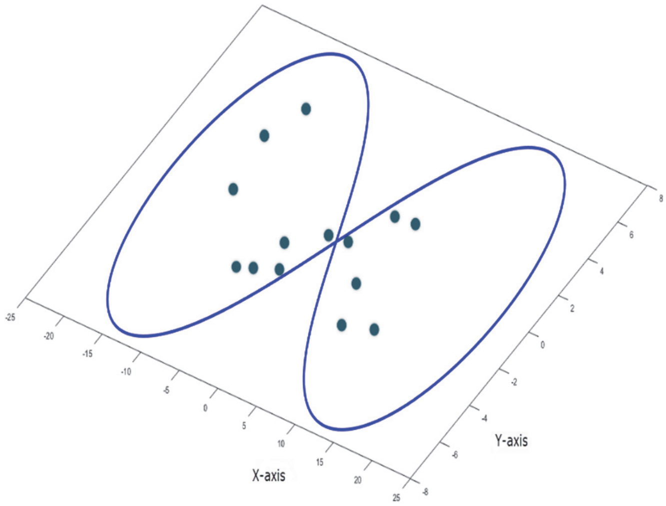

Fig. 9. Floating Cassini ovals.

The n-dimensional situation should also be investigated.

V. Conclusion

Cassini ovals are used both in astronomy and in radiolocation. Cassini ovals are also used in various scientific applications such as physics, biosciences, acoustics, etc. The unique features of ovals make it an interesting tool in various fields for military and commercial purposes.

The study is currently at its initial stage and the author has not rejected the proposed hypothesis of the use of Cassini ovals for clustering purposes and will continue the research.

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