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One-step strong-stability-preserving Hermite-Birkhoff-Taylor methods

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***Abstract* – We construct new one-step explicit multistage strong-stability-preserving (SSP) Hermite–Birkhoff–Taylor (HBT) time discretization methods of orders 3 to 5 for integrating hyperbolic conservation laws. The methods use derivatives y' and y'' as in Taylor methods of order two combined with Runge–Kutta (RK) methods of orders 2 to 4. Compared to RK methods of the same order and with the same number of stages, the new methods generally have larger SSP coefficients on Burgers' equations. Moreover, these SSP HBT methods have stage order two, compared to stage order one for RK methods and hence are less susceptible to order reduction from source terms or nonhomogeneous boundary conditions.**

***Keywords* – Strong stability preserving, Hermite–Birkhoff–Taylor method, SSP coefficient, comparing ODE solvers.**

I. INTRODUCTION

The method of lines is often used to solve time-dependent conservation laws,

$$y_t + g(y)_x = 0, \quad y(x,0) = y_0(x), \tag{1}$$

where the spatial derivative $g(y)_x$ is approximated by a conservative finite difference or finite element at x_j , $j = 1, 2, \ldots N$, (see, for example, [7], [16], [24], [1]). This spatial semi-discretization produces a system of ordinary differential equations with initial conditions of the form

$$y' = f(t,y), \quad y(t_0) = y_0, \text{ where } ' = \frac{d}{dt}. \tag{2}$$

We say that a time discretization method applied to (2) is strong stability preserving (SSP) in a given norm or seminorm $\|\cdot\|$ if the numerical solution $y_{n,j} \approx y(x_j, t_n)$ satisfies the inequality

$$\|y_{n+1}\| \le \|y_n\|. \tag{3}$$

Natural choices are the total variation semi-norm and the maximum norm.

It is assumed that the first-order forward Euler time discretization, FE,

$$y_{n+1} = y_n + \Delta t f(t_n, y_n), \tag{4}$$

and the second-order Taylor time discretization, T2,

$$y_{n+1} = y_n + \Delta t f(t_n, y_n) + \frac{1}{2}\Delta t^2 f'(t_n, y_n), \tag{5}$$

when applied to (2), are SSP for a sufficiently small time step,

$$\Delta t \le \Delta t_{\text{FET2}} = \min\{\Delta t_{\text{FE}}, \Delta t_{\text{T2}}\}, \tag{6}$$

dictated by the Courant–Friedrichs–Levy (CFL) condition [3], [13]. This condition restricts the step size ratio $\Delta t/\Delta x$ of a numerical method applied to a hyperbolic partial differential equation such that the domain of dependence of the numerical solution at the point $P = (x_j, t_n)$ contains the domain of dependence of the exact solution at *P*.

In this paper we construct new one-step explicit multistage SSP Hermite–Birkhoff–Taylor (SSP HBT) time discretization methods to solve system (2).

The objective of high-order SSP HBT time discretizations is to maintain the strong stability property (3) while achieving higher-order accuracy in time, perhaps with a modified CFL restriction (measured here with a SSP coefficient, *c*(HBT)):

$$\Delta t \le c(\text{HBT})\Delta t_{\text{FET2}}. \tag{7}$$

Since our arguments are based on convex decompositions of high-order methods in terms of the first-order Euler method FE and the second-order Taylor method T2, such high-order time discretizations hold for *any* norm and preserve SSP once the FE and T2 time discretizations are shown to be strongly stable, that is,

$$\|y + \Delta t_{\text{FE}} f(t,y)\| \le \|y\| \tag{8}$$

and

$$\|y + \Delta t_{\text{T2}} f(t,y) + (\Delta t_{\text{T2}}^2/2) f'(t,y)\| \le \|y\|, \tag{9}$$

respectively, where $\|\cdot\|$ is a given (semi)norm. Natural choices are the total variation norm and the maximum norm.

A brief review of the development of SSP methods will appear in Section VI on the construction of SSP HBT methods.

We restrict our study to explicit multistage HBT methods with nonnegative coefficients as combinations of T2 and multistage RK methods of orders 2 to 4, incorporating function evaluations at off-step points (see [14], [15]). We have found several new SSP methods with fairly good SSP coefficients by computer search. Compared to known third- to fourthorder SSP RK methods, our new HBT methods of the same order and with the same number of stages per step have larger SSP coefficients. Moreover, these new one-step general linear methods can have stage order two, which is higher than with RK methods, a property that alleviates the order reduction phenomenon encountered in the classical explicit RK methods due to nonhomogeneous boundary/source terms (see [2]). To our knowledge, these are the first SSP one-step explicit multistage methods with stage order two. Furthermore, we have a new one-step HBT method with nonnegative coefficient of order 5, which provides the possibility of constructing new higher-order one-step SSP methods.

Section II introduces two- to five-stage HBT methods. Order conditions of HBT methods of order three to five are listed in Section III. Section IV is concerned with the Shu–Osher representation of SSP HBTs. Section V considers the regions of absolute stability of SSP HBT methods. The construction of SSP HBT methods is considered in Section VI. Section VII validates numerically the order preservation property of SSP HBTs and compares the new methods with SSP RK methods for Burgers’ equation. Appendix A lists the Shu–Osher form of our new SSP HBT methods.

II. TWO – TO FIVE – STAGE HBT’S

In solving (2), one-step *s*-stage HBT methods of order *p*, denoted by HBT*sp*, require s−1 predictors and an integration formula to perform integration from t_n to t_{n+1} (see [14], [15] and references therein). The numbers c_i appearing in the abscissae of the offstep points, $x_n + c_i\Delta t$, are listed in Apppendix A for each new method. In all methods, $c_1 = 0$, but, for simplicity, at times, c_1 will appear in summations. By convention, $c_1^0 = 1$.

Let $F_j := f(t_n + c_j\Delta t, Y_j)$ be the *j*th stage derivatives and set $Y_1 = y_n$ as the initial stage value. Then Hermite–Birkhoff polynomials are used as predictors P_ℓ to obtain the stage values

$$Y_\ell = y_n + \Delta t\sum_{j=1}^{\ell-1} a_{\ell j}F_j + \Delta t^2\gamma_{\ell,2}\, y_n'', \quad \ell = 2,3,\dots,s, \tag{10}$$

to order 2, recursively for $\ell = 2, 3, \dots, s$.

A Hermite–Birkhoff polynomial of degree $s + 1$ is used as integration formula to obtain y_{n+1} to order *p*,

$$y_{n+1} = y_n + \Delta t\sum_{j=1}^{s} b_jF_j + \Delta t^2\gamma_{s+1,2}\,F_1'. \tag{11}$$

One sees that the derivative F_1' is computed only once per step at $t = t_n$. The defining formulae of HBT*sp* involve the usual RK parameters c_i, a_{ij} and b_j and the Taylor expansion parameters $\gamma_{\ell j}$.

All the methods considered in this work are SSP, except the classic 4-stage RK4 of order 4, denoted RK44. Therefore the denomination SSP will often be omitted in what follows.

III. ORDER CONDITIONS OF HBT*SP* MADE OF T2 AND RK*SP*

As in the construction of RK methods, we impose the following simplifying conditions on the abscissa vector $[c_1, c_2, \dots, c_s]^T$:

$$c_i = \sum_{j=1}^{i-1} a_{ij}, \qquad i = 2,3,\dots,s. \tag{12}$$

Forcing an expansion of the numerical solution produced by formulae (10)–(11) to agree with a Taylor expansion of the true solution, we obtain RK-type order conditions that must be satisfied by general HBTsp methods. These order conditions are simply RK order conditions with the Taylor expansion parameters $\gamma_{i,2}$, $i = 2, 3, \dots, s + 1$. To reduce the large number of these RK order conditions to the six sets of order conditions (14)–(19) below, we impose the following simplifying assumptions (where $\gamma_{i,1} = 0$), as in similar searches for ODE solvers [14], [15]:

$$\sum_{j=1}^{i-1} a_{ij}c_j^k + k!\gamma_{i,k+1} = \frac{1}{k+1}c_i^{k+1}, \quad \begin{cases} i = 2,3,\dots,s,\\ k = 0,1.\end{cases} \tag{13}$$

Note that (13) with $k = 0$ is (12).

There remain the following six sets of equations to be solved. Note that $\gamma_{i,1} = 0$ in formula (14).

$$\sum_{i=1}^{s} b_ic_i^k + k!\gamma_{s+1,k+1} = \frac{1}{k+1}, \qquad k = 0,1, \tag{14}$$

$$\sum_{i=1}^{s} b_ic_i^k = \frac{1}{k+1}, \qquad k = 2,3,\dots,p-1, \tag{15}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}\frac{c_j^2}{2!}\right] = \frac{1}{4!}, \tag{16}$$

$$\sum_{i=2}^{s} b_i\frac{c_i}{4}\left[\sum_{j=1}^{i-1} a_{ij}\frac{c_j^2}{2!}\right] = \frac{1}{5!}, \tag{17}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}\frac{c_j^3}{3!}\right] = \frac{1}{5!}, \tag{18}$$

$$\sum_{i=2}^{s} b_i\left[\sum_{j=1}^{i-1} a_{ij}\left(\sum_{k=1}^{j-1} a_{jk}\frac{c_k^2}{2!}\right)\right] = \frac{1}{5!}. \tag{19}$$

- HBT*s*3 made of T2 and an *s*-stage RK2 satisfies (13)–(15) for $s = 2, 3, 4$.
- HBT*s*4 made of T2 and an *s*-stage RK3 satisfies (13)–(16) for $s = 3, 4, 5$.
- HBT55 made of T2 and a 5-stage RK4 satisfies (13)–(19).

IV. SHU – OSHER REPRESENTATION OF SSP HBTS

Following Shu–Osher [21], in this section, we construct HBT methods as convex combinations of FE and T2 methods, which are assumed to be SSP as in (8) and (9), respectively, for sufficiently small Δt satisfying inequality (6).

We first rewrite the explicit HBT method (11)–(12) in the form:

$$Y_1 = y_n,$$

$$Y_i = Y_1 + \Delta t \sum_{k=1}^{i-1} a_{i,k} F_k + \gamma_{i,2} (\Delta t)^2 F_1', \qquad (20)$$

$$i = 2, 3, \dots, s+1,$$

$$y_{n+1} = Y_{s+1},$$

where Y_i is the value of the ith stage at t_n and $F_i := f(t_n + c_i \Delta t, Y_i)$. Note that $a_{s+1,k} = b_k$, $k = 1, 2, \dots, s$, in formula (11).

To have a convex combination, we choose nonnegative numbers α_{ik} such that

$$\alpha_{ik} \geq 0, \qquad \sum_{k=1}^{i-1} \alpha_{ik} = 1.$$

Then, for $i = 2, 3, \dots, s+1$, (20) is transformed into a convex combination of FE and T2 as follows.

$$Y_i = Y_1 + \Delta t \sum_{k=1}^{i-1} a_{i,k} F_k + \gamma_{i,2} (\Delta t)^2 F_1'$$

$$= \left[\sum_{k=1}^{i-1} \alpha_{ik}\right] Y_1 + \Delta t \sum_{k=1}^{i-1} a_{i,k} F_k + \gamma_{i,2} (\Delta t)^2 F_1'$$

$$= \alpha_{i1} Y_1 + \sum_{k=2}^{i-1} \alpha_{ik} \left[Y_k - \Delta t \sum_{\ell=1}^{k-1} a_{k,\ell} F_\ell - \gamma_{k,2} (\Delta t)^2 F_1'\right]$$

$$+ \Delta t \sum_{k=1}^{i-1} a_{i,k} F_k + \gamma_{i,2} (\Delta t)^2 F_1'$$

$$= \alpha_{i1} Y_1 + \sum_{k=2}^{i-1} \alpha_{ik} Y_k - \Delta t \sum_{k=2}^{i-1} \alpha_{ik} \sum_{\ell=1}^{k-1} a_{k,\ell} F_\ell$$

$$- \sum_{k=2}^{i-1} \alpha_{ik} \gamma_{k,2} (\Delta t)^2 F_1' + \Delta t \sum_{k=1}^{i-1} a_{i,k} F_k + \gamma_{i,2} (\Delta t)^2 F_1'$$

$$= \sum_{k=1}^{i-1} \alpha_{ik} Y_k + \Delta t \left[\sum_{k=1}^{i-1} a_{i,k} F_k - \sum_{k=2}^{i-1} \alpha_{ik} \sum_{\ell=1}^{k-1} a_{k,\ell} F_\ell\right]$$

$$+ \left[\gamma_{i,2} - \sum_{k=2}^{i-1} \alpha_{ik} \gamma_{k,2}\right] (\Delta t)^2 F_1'.$$

Since

$$\sum_{k=2}^{i-1} \alpha_{ik} \sum_{\ell=1}^{k-1} a_{k,\ell} F_\ell = \sum_{k=1}^{i-1} \sum_{\ell=k+1}^{i-1} \alpha_{i,\ell} a_{\ell,k} F_k,$$

we have

$$Y_i = \sum_{k=1}^{i-1} \left[\alpha_{ik} Y_k + \left[a_{i,k} - \sum_{\ell=k+1}^{i-1} \alpha_{i,\ell} a_{\ell,k}\right] \Delta t F_k\right]$$

$$+ \delta_i (\Delta t)^2 F_1', \qquad (21)$$

where

$$\delta_i = \gamma_{i,2} - \sum_{\ell=2}^{i-1} \alpha_{i,\ell}\, \gamma_{\ell,2}.$$

If in (21) we let

$$\beta_{ik} = a_{i,k} - \sum_{\ell=k+1}^{i-1} \alpha_{i,\ell} a_{\ell,k},$$

we can rewrite (20) (or (21)) in the Shu–Osher equivalent form, where Y_i, are now convex combinations of forward Euler and T2 solvers with suitably scaled Δt's,

$$Y_i = \sum_{k=1}^{i-1} [\alpha_{ik} Y_k + \beta_{ik} \Delta t F_k)] + \delta_i (\Delta t)^2 F_1'$$

$$= \alpha_{i1} \left[Y_1 + \frac{\beta_{i1}}{\alpha_{i1}} \Delta t F_1 + \frac{\delta_i}{\alpha_{i1}} (\Delta t)^2 F_1'\right]$$

$$+ \sum_{k=2}^{i-1} \alpha_{ik} \left[Y_k + \frac{\beta_{ik}}{\alpha_{ik}} \Delta t F_k\right]. \qquad (22)$$

To get the term $\left[Y_1 + \frac{\beta_{i1}}{\alpha_{i1}} \Delta t F_1 + \frac{\delta_i}{\alpha_{i1}} (\Delta t)^2 F_1'\right]$ in (22) as integration step (4) of T2, we need

$$\frac{1}{2} \left(\frac{\beta_{i1}}{\alpha_{i1}}\right)^2 = \frac{\delta_i}{\alpha_{i1}}, \quad i = 2, 3, \dots, s+1. \qquad (23)$$

Our computer search has found no HBT methods satisfying condition (23).

Now we replace the restrictive formula

$$\alpha_{i1} \left[Y_1 + \frac{\beta_{i1}}{\alpha_{i1}} \Delta t F_1 + \frac{\delta_i}{\alpha_{i1}} (\Delta t)^2 F_1'\right]$$

in (22) with the less restrictive convex combination of forward Euler and T2 formulae:

$$\alpha_{i1}\mu_{i1}\left[Y_1 + \frac{\beta_{i1}\nu_{i1}}{\alpha_{i1}\mu_{i1}}\Delta t F_1 + \frac{\delta_i}{\alpha_{i1}\mu_{i1}}(\Delta t)^2 F_1'\right] + \alpha_{i1}\mu_{i2}\left[Y_1 + \frac{\beta_{i1}\nu_{i2}}{\alpha_{i1}\mu_{i2}}\Delta t F_1\right],$$

where $\mu_{i1}+\mu_{i2}=1$ and $\nu_{i1}+\nu_{i2}=1$ with nonnegative μ_{i1}, μ_{i2}, ν_{i1} and ν_{i2}, and rewrite (22) as

$$Y_i = \alpha_{i1}\mu_{i1}\left[Y_1 + \frac{\beta_{i1}\nu_{i1}}{\alpha_{i1}\mu_{i1}}\Delta t F_1 + \frac{\delta_i}{\alpha_{i1}\mu_{i1}}(\Delta t)^2 F_1'\right] + \alpha_{i1}\mu_{i2}\left[Y_1 + \frac{\beta_{i1}\nu_{i2}}{\alpha_{i1}\mu_{i2}}\Delta t F_1\right] + \sum_{k=2}^{i-1}\alpha_{ik}\left[Y_k + \frac{\beta_{ik}}{\alpha_{ik}}\Delta t F_k\right],$$

$$i = 2,3,\ldots,s+1. \qquad (24)$$

Hence, (24) is a convex combination of

- T2 with step size $\frac{\beta_{i1}\nu_{i1}}{\alpha_{i1}\mu_{i1}}\Delta t$ such that

$$\frac{1}{2}\left[\frac{\beta_{i1}\nu_{i1}}{\alpha_{i1}\mu_{i1}}\right]^2 = \frac{\delta_i}{\alpha_{i1}\mu_{i1}} \qquad (25)$$

  if $\delta_i > 0$, and of

- FE steps with step sizes $\frac{\beta_{i1}\nu_{i2}}{\alpha_{i1}\mu_{i2}}\Delta t$ and $\frac{\beta_{ik}}{\alpha_{ik}}\Delta t$ for $k \geq 2$.

HBT methods, when written in the form (24), can conveniently use the result of the following theorem, which is a straightforward extension of a corresponding result presented in [21], [6].

Theorem 1: If the FE and T2 methods are strongly stable under the CFL restriction $\Delta t \leq \Delta t_{\mathrm{FET2}} = \min\{\Delta t_{\mathrm{FE}}, \Delta t_{\mathrm{T2}}\}$, then the HBT method (22), with $\beta_{ik} \geq 0$ and $\delta_i \geq 0$, is SSP provided

$$\Delta t \leq c(\mathrm{HBT})\Delta t_{\mathrm{FET2}},$$

where c(HBT) is the SSP coefficient

$$c(\mathrm{HBT}) = \min\left\{\min_{\substack{i=2,\ldots,s+1\\k=2,\ldots,i-1}}\left\{\frac{\alpha_{ik}}{\beta_{ik}}\right\}, \min_{i=2,\ldots,s+1}\left\{\frac{\alpha_{i1}\mu_{i2}}{\beta_{i1}\nu_{i2}}, \frac{\alpha_{i1}\mu_{i1}}{\beta_{i1}\nu_{i1}}\right\}\right\}, \qquad (26)$$

such that (25) holds, with the convention that $a / 0 = +\infty$.

This theorem provides a theoretical criterion for optimizing a given HBT method. As pointed out by Gottlieb [4], one looks for high-order SSP methods with c(HBT) as large as possible, taking their computational costs and order into account.

Definition 1 (See [19], [23]): The *effective SSP coefficients* of HBT and RK methods are

$$c_{\mathrm{eff}}(\mathrm{HBT}) = \frac{\mathrm{c(HBT)}}{\ell}, \quad c_{\mathrm{eff}}(\mathrm{RK}) = \frac{\mathrm{c(RK)}}{\ell}, \qquad (27)$$

respectively, where ℓ is the number of function evaluations per time step and c(RK) is defined in [22].

In this paper, ℓ = 6, 5, 4, 3 for (ℓ − 1)-stage HBT methods. When the context is clear, for short, we write c and c_{eff}.

The effective SSP coefficient provides a fair comparison between methods of the same order.

V. REGIONS OF ABSOLUTE STABILITY

To obtain the region of absolute stability, *R*, of HBT methods, we apply the predictors P_i (10) with $i = 2, \ldots, s$, and integration formula(11) with constant time step Δt to the linear test equation

$$y' = \lambda y, \quad y_0 = 1.$$

Thus, we obtain

$$Y_\ell = y_n + \lambda\Delta t\sum_{j=1}^{\ell-1}a_{\ell j}Y_j + (\lambda\Delta t)^2\gamma_{\ell,2}\,y_n,$$

$$\ell = 2,3,\ldots,s, \qquad (28)$$

and

$$y_{n+1} = y_n + \lambda\Delta t\sum_{j=1}^{s}b_jY_j + \sum_{j=2}^{2}(\lambda\Delta t)^j\gamma_j\,y_n. \qquad (29)$$

Since $Y_1 = y_n$, it is seen that Y_2 in (28) is expressed in terms of y_n only. Therefore, proceeding recursively, we can express Y_ℓ, ℓ = 3, . . . , s, in terms of y_n only. Finally, y_{n+1} in (29) is expressed in terms of y_n only. Thus, we obtain the following first-order difference equation and the associated linear characteristic equation:

$$-r_s y_n + y_{n+1} = 0, \qquad r = r_s. \qquad (30)$$

If s is the number of stages, the coefficients s_j in the root

$$r_s = 1 + \sum_{j=1}^{s+1}s_j\lambda^j\Delta t^j,$$

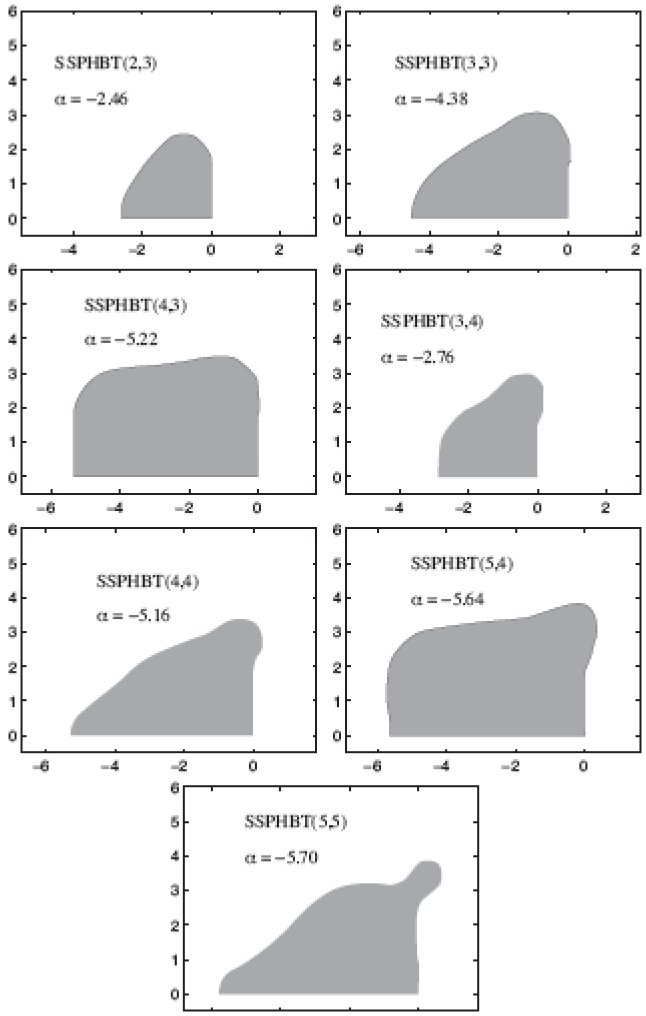


Fig. 1. Regions of absolute stability of HBTs

can be found by symbolic computation.

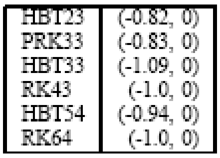
A complex number $\lambda\Delta t$ is in *R* if r_s satisfies the root condition, that is, $|r_s| \leq 1$ (see [12, pp. 70]).

The root condition is used to find the regions of absolute stability of HBT methods whose upper parts are shown in grey in Fig. 1. The abscissae α of the interval of absolute stability, (α, 0), are also indicated in the figure.

In Table I, we note that scaled intervals of absolute stability of HBT methods are similar to approximate scaled intervals of absolute stability of RK methods read from the figures in [10].

We remark that the regions of stability contain a segment of the imaginary axis, which is a desirable property when solving PDEs via the method of lines with certain spatial discretizations [9]. The regions of linear stability provide valuable insights for the behavior of numerical methods when applied to nonlinear systems. The stability regions of the new HBT methods are useful in the solution of ODEs, independently of their SSP properties which are needed in the solution of conservation laws.

TABLE I
SCALED INTERVALS OF ABSOLUTE STABILITY OF HBT METHODS VERSUS APPROXIMATE SCALED INTERVALS OF SSP RK METHODS.



HBT23	(-0.82, 0)
PRK33	(-0.83, 0)
HBT33	(-1.09, 0)
RK43	(-1.0, 0)
HBT54	(-0.94, 0)
RK64	(-1.0, 0)

VI. CONSTRUCTION OF HBT METHODS

Since HBT methods depend on several free parameters, we used the Matlab Optimization Toolbox to search for methods with largest *c*(HBT) to an accuracy of 10^{-12} for different number of stages, *s*. The tolerance on the constraints, including the order conditions, was 10^{-14}. The required number of memory registers that have to be stored after each step for each new method is determined as in [2]. For each new method, we list, in Appendix A, the SSP coefficient $c = c(\text{HBT})$, the effective SSP coefficient $c_{\text{eff}} = c_{\text{eff}}(\text{HBT})$, the required number of memory registers, the abscissa vector $[c_1, c_2, \ldots]^T$ and the method in the Shu–Osher form.

A. Third-order methods

The most popular SSP time discretization method is the optimal three-stage RK33 method of order 3, given in Gottlieb, Shu and Tadmor [6] with $c(\text{RK})=1$ and $c_{\text{eff}}(\text{RK})=1/3$. Spiteri and Ruuth [22] found some new methods with more stages. For example, the optimal four-stage method, RK43, has $c(\text{RK}) = 2$ and $c_{\text{eff}}(\text{RK}) = 0.5$.

We present third-order HBT methods with 2, 3 and 4 stages. These methods combine T2 and RK2. If we further increase the number of stages, we can find HBT methods with larger *c*(HBT).

B. Fourth-order methods

Gottlieb and Shu [5] have proved that there are no fourstage, fourth-order SSP methods with nonnegative coefficients. Spiteri and Ruuth [22] found a five-stage SSP RK method of order 4, RK54, with $c(\text{RK})=1.508$ and $c_{\text{eff}}(\text{RK})=0.302$ (see [11] for another independent proof in contractivity studies). Other fourth-order SSP RK methods with more stages are found in Spiteri and Ruuth [23]. As in the case of RK methods, increasing the number of stages of HBT methods improves the effective SSP coefficient.

We construct fourth order HBT's with 3, 4 and 5 stages and nonnegative coefficients by combining T2 and RK3.

C. Fifth-order methods

Ruuth and Spiteri [18] proved that there are no fifth-order SSP RK methods with nonnegative coefficients (they recently considered fifth-order methods with negative coefficients in [17], [19]). We have not found in the literature any one-step fifth-order SSP method with nonnegative coefficients. But, our investigation for one-step HBT methods shows that one such

HBT with nonnegative coefficients not only exist, but also has a fairly good SSP coefficient. For example, combining T2 with a five-stage SSP RK method of order 4, we obtain HBT55 with nonnegative coefficients of order 5 with 5 stages and *c*(HBT) = 1.062.

VII. NUMERICAL RESULTS

From now on, we use the total variation semi-norm,

$$\mathrm{TV}(y_n) = \sum_j |y_{n,j+1} - y_{n,j}|, \tag{31}$$

and say that a method is total variation diminishing (TVD) if

$$\mathrm{TV}(y_{n+1}) \leq \mathrm{TV}(y_n). \tag{32}$$

A. Validating the order preservation

To illustrate the order-reduction phenomenon due to boundary or source terms we consider a classical initial-boundary value test problem with a nonlinear source described in [20]:

$$\begin{cases} u_t(x,t) = -u_x(x,t) + b(x,t), & \begin{cases} 0 \leq x \leq 1, \\ 0 \leq t \leq 1, \end{cases} \\ u(x,0) = 1 + x, \\ u(0,t) = 1/(1+t). \end{cases} \tag{33}$$

The source term is $b(x,t) = (t-x)/(1+t)^2$. Since the exact solution, $u(x,t) = (1+x)/(1+t)$, is linear in *x*, we can use first-order upwind space discretization without introducing discretization errors. The SSP RK54 and the non-SSP classic RK44 are used for integration in time. All explicit RK methods considered here have stage order equal to one. Sanz Serna *et al.* [20] show that explicit RK methods with $p \geq 3$ suffer from order reduction on problems with nonhomogeneous boundary conditions or nonzero source terms such as (33).

For problem (33), we distinguish two cases, one that illustrates the order reduction phenomenon, and, for validation purposes, one that does not. Specifically, if the spatial and temporal grids are refined simultaneously, one notices that low stage order methods suffer from order reduction. If the space grid is maintained fixed, that is, the ODE problem is fixed, then the (classical) order of consistency is preserved.

In the cases of RK44 and RK54, Table II shows the discretization error versus the time step without order reduction when $\Delta x = 1/10$ and with order reduction when $\Delta x = 1/20$. It is seen that HBT44 and HBT54 maintain well their consistency orders. Moreover, the observed errors are much smaller than for the RK methods with stage order one, under consideration here.

B. HBTsp vs. RKsp on Burgers' equation with unit-step initial condition

For a first comparison of our new HBT*sp* with RKsp, following Huang [8], we consider Burgers' equation in Problem 1.

TABLE II
L∞-ERROR OF HBT44, RK44, HBT54 AND RK54 AT *t* = 1 FOR PROBLEM (33).

Δt	Δx		Δx	
	1/10	1/20	1/10	1/20
	HBT44		RK44	
1/20	9.13e-8	1.28e-7	2.62e-6	1.63e-5
1/40	5.52e-9	7.74e-9	1.27e-7	6.55e-7
1/80	3.39e-10	4.76e-10	6.91e-9	3.24e-8
	HBT54		RK54	
1/20	4.88e-8	6.77e-8	1.14e-6	5.89e-6
1/40	2.96e-9	4.09e-9	6.08e-8	2.89e-7
1/80	1.82e-10	2.51e-10	3.47e-9	1.56e-8

Problem 1: Burgers' equation with unit-step initial condition:

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) + \dfrac{\partial}{\partial x}\left[\dfrac{1}{2} u(x,t)^2\right] = 0, \\ u(x,0) = \begin{cases} 1, & -1 \leq x < 0, \\ 0, & 0 < x \leq 1, \end{cases} \\ u(-1,t) = 1, \quad \text{for} \quad t \geq 0. \end{cases} \tag{34}$$

We discretize the spatial derivative by the difference quotient

$$\frac{1}{\Delta x}\left[\frac{1}{2}(u_j(t))^2 - \frac{1}{2}(u_{j-1}(t))^2\right] \tag{35}$$

with space stepsize $\Delta x = 1/150$, where $u_j(t)$ is an approximation to $u(x_j, t)$ with $x_j = j\Delta x$,
j = . . . ,−2,−1, 0, 1, 2, ... This leads to the semi-discrete system

$$\frac{d}{dt} u_j(t) = -\frac{1}{\Delta x}\left[\frac{1}{2}(u_j(t))^2 - \frac{1}{2}(u_{j-1}(t))^2\right],$$

to which a time discretization can be applied.

We consider the total variation seminorm of the numerical solution at $t_{\text{final}} = 1.8$. For this purpose, we let num_{eff} be the *largest effective CFL number* defined as

$$\text{num}_{\text{eff}} = \max_{\Delta t}\left\{\frac{1}{l}\frac{\Delta t}{\Delta x}\right\}, \tag{36}$$

such that the TV error in the numerical solution satisfies the inequality

$$|\text{TV of } u(x, t_{\text{final}}) - \text{TV of } u(x,0)| \leq 0.05, \tag{37}$$

and we let $\max \Delta t_{\text{num}} = l\Delta x$: num_{eff} be the maximum numerical step size. Here *l* is the number of function evaluations per time step. Finally, we let $\max \Delta t_{\text{theor}}$ of HBT for Problem 1 be taken as

$$\max \Delta t_{\text{theor}} = c(\text{HBT})\Delta t_{\text{FET2}}, \tag{38}$$

where *c*(HBT) are the SSP coefficients of HBTs listed in Appendix A and $\Delta t_{\text{FET2}} = \min\{\Delta t_{\text{FE}}, \Delta t_{\text{T2}}\}$ defined in (7).

It was observed numerically that the TVD property (5) holds with error (37) for the methods listed in Table III for $\Delta t \le \max \Delta t_{\text{num}}$. More explicitly, the combination of the difference quotient (35) with FE and T2 satisfies the TVD property (5). Since the HBT methods described in this work are convex combinations of FE and T2, as proved in Theorem 1, the combination of (35) with the HBT also is TVD. The same holds for the combination of (35) with the listed RK methods, the latter being expressed as convex combinations of FE [21]. The same situation holds for Problem 2 below.

TABLE III
$\text{MN}=\max\Delta t_{\text{NUM}}/\Delta x$ AND $\text{MT}=\max\Delta t_{\text{THEOR}}/\Delta x$ OF HBT FOR PROBLEMS 1 (LEFT) AND 2 (RIGHT) WITH ERROR (37).

Method	MN	MT	MN	MT
HBT23	1.646	1.092	1.252	1.016
HBT33	2.592	1.872	2.044	1.741
HBT43	2.880	2.549	2.629	2.371
HBT34	1.650	1.092	1.384	1.016
HBT44	2.535	1.985	2.193	1.847
HBT54	3.074	2.665	2.800	2.480
HBT55	2.676	1.134	2.302	1.055
T2	1.092	1.092	1.008	1.016
FE	1.166	1.166	1.132	1.188
RK33	1.496	1.166	1.244	1.188
RK43	2.256	2.332	2.040	2.376
RK54	2.472	1.758	2.188	1.791

We observe in columns two and three of Table III that, with the same number of stages and order, HBTs have larger maximum numerical and theoretical step sizes than RK methods. We also observe that $\max \Delta t_{\text{num}} > \max \Delta t_{\text{theor}}$, except for RK43. The theoretical strong stability bounds are thus generally verified in this numerical comparison of maximum time steps.

We note that FE has SSP coefficient equal to the CFL condition for Burgers' equation. Hence, with $\Delta t = \Delta x = 1/150$, the TV error (37) is zero.

C. HBTsp vs. RKsp on Burgers' equation with square-wave initial condition

We now consider Burgers' equation with a square-wave initial value in Problem 2, which is Laney's Test Case 4 [13, p. 312].

Problem 2: Burgers' equation with a square wave initial condition:

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) + \dfrac{\partial}{\partial x}\left[\dfrac{1}{2}u(x,t)^2\right] = 0, \\ u(x,0) = \begin{cases} 1, & |x| \le \frac{1}{3}, \\ 0, & \frac{1}{3} < |x| \le 1, \end{cases} \\ u(-1,t) = 0, \quad \text{for} \quad t \ge 0. \end{cases} \tag{39}$$

We discretize the spatial derivative by the difference quotient (35) and compute the total variation of the numerical solution of Problem 2 as a function of the *CFL number* $\Delta t/\Delta x$ at $t = 0.6$.

We calculate the maximum theoretical time step of HBT for Problem 2 using (38). Then we can compare $\max\{\Delta t_{\text{theor}}/\Delta x\}$ with $\max\{\Delta t_{\text{num}}/\Delta x\}$ of HBT methods listed in last two columns of Table III. We again observe that, with the same number of stages and order, HBT methods have larger maximum numerical and theoretical step size than corresponding RK methods. As with RK methods, $\max\{\Delta t_{\text{num}}\} > \max\{\Delta t_{\text{theor}}\}$ with HBT methods. This comparison of numerical maximum time steps in Problem 2 confirms again that the theoretical strong stability bounds are verified.

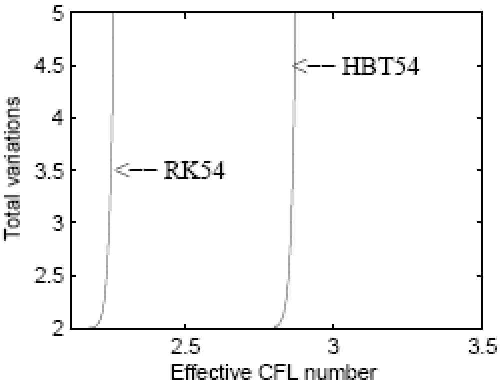


Fig. 2. Total variation of the solution as a function of $\mathrm{num}_{\mathrm{eff}}$ for HBT54 and RK54 applied to Problem 2

Figure 2 shows that the total variation of the solution of HBT54 for Problem 2 compares favorably with that of RK54 as a function of $\mathrm{num}_{\mathrm{eff}}$ while these methods have the same number of stages and same order.

VIII. CONCLUSION

Several new multi-stage SSP HBT explicit methods of orders 3 to 5 with nonnegative coefficients were constructed. These methods combine Taylor method of order 2 and RK methods of orders 2 to 4. Compared to RK methods of the same order, and with the same number of stages, the new methods, generally, have larger SSP coefficients, and larger CFL numbers, or larger maximum time steps, on Burgers' equation. These new one-step methods have stage order two which is higher than stage order one for the RK methods considered here, and, hence, are less susceptible than RK methods to order reduction from source terms or nonhomogeneous boundary conditions. To our knowledge, these are the first SSP one-step explicit multistage methods with stage order two. Moreover, we have constructed a new one-step SSP HBT method of order 5 with nonnegative coefficients. SSP HBT methods with second derivatives y'' appear to be promising for integrating hyperbolic conservation laws in the light of our numerical results since one-step SSP methods of high-order can be derived and implemented efficiently.

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APPENDIX

This appendix contains our new s-stage HBT*sp* methods of order *p* in Shu–Osher form. For each method, *c* = *c*(HBT), $c_{\text{eff}} = c_{\text{eff}}(\text{HBT})$, the number of required registers that need to be stored after each stage, and the abscissa vector $[c_1, c_2, \dots]^T$ are listed. We use $Y_1 = y_n$ and $Y_{s+1} = y_{n+1}$.

As an example, for HBT33 below, the number of required registers is the maximum number of elements $\dots, Y_1, F_1, F_1', \dots$ listed in square brackets [ ], namely, 5.

- For Y_2, $[Y_2, F_2, Y_1, F_1, F_1']$,
- For Y_3, $[Y_3, F_3, Y_1, F_1, F_1']$,
- For y_{n+1}, $[y_{n+1}, Y_1, F_1, F_1']$.

A. HBTs3 made of T2 and RK2

HBT23 *c* = 1.0, c_{eff} = 1/3; 5 registers and $[0.0, 1.0]^T$.

$$Y_2 = Y_1 + \Delta t F_1 + \frac{1}{2}(\Delta t)^2 F_1',$$
$$y_{n+1} = \frac{2}{3} Y_1 + \frac{1}{3}\Delta t F_1 + \frac{1}{3} Y_2 + \frac{1}{3}\Delta t F_2.$$

HBT33 *c* = 1.714, c_{eff} = 0.428; 5 registers and $[0.0, 0.5833333333333333, 0.875]^T$.

$$\begin{aligned}
Y_2 &= Y_1 + 5.833333333333333\,\text{e-01}\Delta t F_1 \\
&\quad + 1.701388888888888\,\text{e-01}(\Delta t)^2 F_1', \\
Y_3 &= 2.50\,\text{e-01}Y_1 + 7.5\,\text{e-01}Y_2 + 4.375\,\text{e-01}\Delta t F_2, \\
y_{n+1} &= 4.402332361516034\,8\,\text{e-01}Y_1 \\
&\quad + 1.836734693877551\,4\,\text{e-01}\Delta t F_1 \\
&\quad + 5.597667638483965\,2\,\text{e-01}Y_3 \\
&\quad + 3.265306122448979\,3\,\text{e-01}\Delta t F_3.
\end{aligned}$$

HBT43 *c* = 2.414, c_{eff} = 0.483; 5 registers and $[0.0, 0.41431034325797217, 0.62114634125991519, 0.86968146363925003]^T$.

$$\begin{aligned}
Y_2 &= Y_1 + 4.1431034325797217\,\text{e-01}\Delta t F_1 \\
&\quad + 8.5826530265269360\,\text{e-02}(\Delta t)^2 F_1', \\
Y_3 &= 2.4923022017337446\,\text{e-01}Y_1 \\
&\quad + 7.5076977982662552\,\text{e-01}Y_2 \\
&\quad + 3.1009465607223380\,\text{e-01}\Delta t F_2, \\
Y_4 &= 1.6009865328347142\,\text{e-01}Y_1 \\
&\quad + 2.4785663469860693\,\text{e-10}Y_2 \\
&\quad + 8.3990134646867198\,\text{e-01}Y_3 \\
&\quad + 3.4797981515826842\,\text{e-01}\Delta t F_3, \\
y_{n+1} &= 3.2294363363947282\,\text{e-01}Y_1 \\
&\quad + 1.3066517278547909\,\text{e-01}\Delta t F_1 \\
&\quad + 9.2180652000450901\,\text{e-10}(\Delta t)^2 F_1' \\
&\quad + 6.7705636636052724\,\text{e-01}Y_4 \\
&\quad + 2.8051145555182538\,\text{e-01}\Delta t F_4.
\end{aligned}$$

B. HBTs4 made of T2 and RK3

HBT34 *c* = 1.0, c_{eff} = 1/4; 5 registers and $[0.0, 1.0, 2/3]^T$.

$$\begin{aligned}
Y_2 &= Y_1 + \Delta t F_1 + 5.0\,\text{e-01}(\Delta t)^2 F_1', \\
Y_3 &= 8.5185185185185185\,\text{e-01}Y_1 \\
&\quad + 3.7037037037037037\,\text{e-01}\Delta t F_1 \\
&\quad + 1.4814814814814814\,\text{e-01}Y_2 \\
&\quad + 1.4814814814814814\,\text{e-01}\Delta t F_2, \\
y_{n+1} &= 4.3750000000000000\,\text{e-01}Y_1 \\
&\quad + 6.2500000000000000\,\text{e-02}\Delta t F_1 \\
&\quad + 5.6249999999999999\,\text{e-01}Y_3 \\
&\quad + 5.6249999999999999\,\text{e-01}\Delta t F_3.
\end{aligned}$$

HBT44 *c* = 1.818, c_{eff} = 0.363; 5 registers and $[0.0, 0.55000000009635985, 0.82500000014453978, 0.74642857135384333]^T$.

$$\begin{aligned}
Y_2 &= Y_1 + 5.5000000009635985\,\text{e-01}\Delta t F_1 \\
&\quad + 1.5125000005299793\,\text{e-01}(\Delta t)^2 F_1', \\
Y_3 &= 2.5\,\text{e-01}Y_1 + 7.5\,\text{e-01}Y_2 \\
&\quad + 4.1250000007226983\,\text{e-01}\Delta t F_2, \\
Y_4 &= 6.4917395558616608\,\text{e-01}Y_1 \\
&\quad + 2.6404276020030792\,\text{e-01}\Delta t F_1 \\
&\quad + 7.6696343720428204\,\text{e-11}(\Delta t)^2 F_1' \\
&\quad + 3.5082604441383386\,\text{e-01}Y_3 \\
&\quad + 1.9295432446141420\,\text{e-01}\Delta t F_3, \\
y_{n+1} &= 1.5744125555737143\,\text{e-01}Y_1 \\
&\quad + 1.9899002824857098\,\text{e-02}\Delta t F_1 \\
&\quad + 2.4245385850045487\,\text{e-04}(\Delta t)^2 F_1' \\
&\quad + 1.5033753603771832\,\text{e-01}Y_2 \\
&\quad + 6.9222120840491030\,\text{e-01}Y_4 \\
&\quad + 3.8072166468940305\,\text{e-01}\Delta t F_4.
\end{aligned}$$

HBT54 *c* = 2.441, c_{eff} = 0.407; 5 registers and $[0.0, 0.40974441615268270, 0.61461661286849734, 0.75157694893948479, 0.79830427988647235]^T$.

$$\begin{aligned}
Y_2 &= Y_1 + 4.0974441615268270\,\text{e-01}\Delta t F_1 \\
&\quad + 8.3945243284151413\,\text{e-02}(\Delta t)^2 F_1', \\
Y_3 &= 2.4999997227411563\,\text{e-01}Y_1 \\
&\quad + 7.5000002772588437\,\text{e-01}Y_2 \\
&\quad + 3.0730828939345894\,\text{e-01}\Delta t F_2,
\end{aligned}$$

$$\begin{aligned}
Y_4 = {} & 3.5914242260746471 \text{ e-}01 Y_1 \\
& + 9.5107421505646939 \text{ e-}02 \Delta t F_1 \\
& + 1.1713761904896813 \text{ e-}11 (\Delta t)^2 F_1' \\
& + 6.4085757739253524 \text{ e-}01 Y_3 \\
& + 2.6258781388572700 \text{ e-}01 \Delta t F_3, \\
Y_5 = {} & 4.3671737051794879 \text{ e-}01 Y_1 \\
& + 1.7761161908152462 \text{ e-}01 \Delta t F_1 \\
& + 8.6356923799703056 \text{ e-}03 (\Delta t)^2 F_1' \\
& + 4.4519049506596256 \text{ e-}02 Y_2 \\
& + 5.1876357997545486 \text{ e-}01 Y_4 \\
& + 2.1256048019831827 \text{ e-}01 \Delta t F_4, \\
y_{n+1} = {} & 3.2632877500399101 \text{ e-}03 Y_1 \\
& + 2.5567504856286760 \text{ e-}01 Y_2 \\
& + 7.4106166368709259 \text{ e-}01 Y_5 \\
& + 3.0364587872060345 \text{ e-}01 \Delta t F_5.
\end{aligned}$$

C. HBT55 made of T2 and RK4

HBT55 $c = 1.062$, $c_{\text{eff}} = 0.177$; 5 registers and
$$[0.0, 0.42850362365776878, 0.65248338990632693, \\ 0.60260786609403039, 0.94283575659205909]^T.$$

$$\begin{aligned}
Y_2 = {} & Y_1 + 4.2850362365776878 \text{ e-}01 \Delta t F_1 \\
& + 9.1807677743919375 \text{ e-}02 (\Delta t)^2 F_1', \\
Y_3 = {} & 5.7004528299657609 \text{ e-}01 Y_1 \\
& + 6.3595881789371367 \text{ e-}02 \Delta t F_1 \\
& + 4.2995471700342403 \text{ e-}01 Y_2 \\
& + 4.0465035387223786 \text{ e-}01 \Delta t F_2, \\
Y_4 = {} & 7.8043632180052036 \text{ e-}01 Y_1 \\
& + 2.5270463762198825 \text{ e-}01 \Delta t F_1 \\
& + 2.1956367819947972 \text{ e-}01 Y_3 \\
& + 2.0664157542014377 \text{ e-}01 \Delta t F_3, \\
Y_5 = {} & 4.0635304549864582 \text{ e-}01 Y_1 \\
& + 2.6390710686616886 \text{ e-}02 \Delta t F_1 \\
& + 3.9083207156726945 \text{ e-}12 (\Delta t)^2 F_1' \\
& + 5.9364695450135430 \text{ e-}01 Y_4 \\
& + 5.5870872144016104 \text{ e-}01 \Delta t F_4, \\
y_{n+1} = {} & 3.7617138439580072 \text{ e-}01 Y_1 \\
& + 6.2512901582617034 \text{ e-}02 \Delta t F_1 \\
& + 1.6161212809706466 \text{ e-}01 Y_2 \\
& + 1.5210067999789351 \text{ e-}01 \Delta t F_2 \\
& + 7.0070576175738281 \text{ e-}02 Y_3 \\
& + 2.0098248882263273 \text{ e-}01 Y_4 \\
& + 1.8915395507465443 \text{ e-}01 \Delta t F_4 \\
& + 1.9116342250876373 \text{ e-}01 Y_5 \\
& + 1.7991185494691220 \text{ e-}01 \Delta t F_5.
\end{aligned}$$

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Truongs Nguen-Ba, Huong Nguen-Thu, Thierry Giordanos, Remi Vajekurs. Ermita-Birkhofa-Teilora viensoļu metodes, kas saglabā stingru stabilitāti

Vienādojumu, kuri atkarīgi no laika un pierakstīti konservatīvā formā, atrisināšanai bieži izmanto taišņu metodi. Šādu vienādojumu telpiska diskretizācija noved pie parasto diferenciālvienādojumu sistēmas. Tiek pieņemts, ka diskretizācija pēc laika, veikta ar pirmās kārtas Eilera metodes un otrās kārtas Teilora metodes palīdzību, saglabā stingras stabilitātes īpašību (SSSĪ) pie pietiekami maza laika soļa, kuru nosaka ar Kuranta-Fridriksa-Levi nosacījuma palīdzību. Mērķis augstākas kārtas Ermita-Birkhofa-Teilora metodēm, kas saglabā stingras stabilitātes īpašību, ir saglabāt augstas kārtas precizitāti pēc laika, izmantojot modificēto Kuranta-Fridriksa-Levi nosacījumu, kuru raksturo stingras stabilitātes koeficients. Metožu analīze rakstā ir ierobežota ar atklātu daudzsoļu Ermita-Birkhofa-Teilora metožu aplūkošanu, kurās izmantoti nenegatīvi koeficienti un funkcijas vērtības punktos, kas atrodas ārpus tīkla. Ar datora meklēšanas palīdzību autoriem izdevās atrast dažas jaunas SSSĪ metodes ar pietiekami labiem SSSĪ koeficientiem. Iegūtās metodes satur lielus SSSĪ koeficientus salīdzinājumā ar tādas pašas kārtas zināmām metodēm ar to pašu soļu skaitu. Rakstā ir konstruēta jauna viensoļa Ermita-Birkhofa-Teilora metode ar kārtu 5 ar nenegatīviem koeficientiem, kas dod iespēju konstruēt jaunas augstākas kārtas SSSĪ metodes.

Труонг Нгуен-Ба, Хуонг Нгуен-Ху, Тьерри Джордано, Реми Вайенкур. Одношаговые методы Эрмита-Биркхофа-Тейлора, сохраняющие сильную устойчивость

Метод прямых часто используется для решения зависящих от времени уравнений, записанных в консервативной форме. Пространственная дискретизация таких уравнений приводит к системе обыкновенных дифференциальных уравнений. Предполагается, что дискретизация по времени с помощью метода Эйлера первого порядка и метода Тейлора второго порядка сохраняют свойство сильной устойчивости (СССУ) для достаточно малого шага по времени, который определяется с помощью условия Куранта-Фридрикса-Леви. Цель методов Эрмита-Биркхофа-Тейлора высокого порядка, сохраняющих свойство сильной устойчивости, заключается в достижении высокого порядка точности по времени с использованием модифицированного условия Куранта-Фридрикса-Леви, которое характеризуется коэффициентом сильной устойчивости (КСУ). Анализ методов в статье ограничен явными многошаговыми методами Эрмита - Биркхофа - Тейлора с неотрицательными коэффициентами с использованием вычисления функций в точках, лежащих вне сетки. С помощью компьютерного поиска нам удалось найти несколько новых СССУ методов с достаточно хорошими СССУ коэффициентами. Полученные методы имеют большие значения СССУ коэффициентов по сравнению с известными методами того же порядка с тем же числом шагов. В статье также предложен новый одношаговый метод Эрмита - Биркхофа – Тейлора порядка 5 с неотрицательными коэффициентами, что позволяет построить новые одношаговые СССУ методы высокого порядка.